

CAPACITY ESTIMATES FOR TRO CHANNELS

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ABSTRACT. Using the notion of TRO's (ternary ring of operators) and independence from operator algebra theory, we discover a new class of channels which allow single letter bounds for their quantum and private capacity, as well as strong converse rates. This class goes beyond degradable channels. The estimate are based on a "local comparison theorem" for sandwiched Rényi relative entropy and complex interpolation. As an application, we discover new small dimensional examples which admit an easy formula for quantum and private capacity.

1. INTRODUCTION

During the last decades, Shannon's theory on transmitting information via noisy channels has been adapted to quantum media. Various capacities are introduced to describe different tasks of a quantum channel. These capacities, such as quantum capacity and private capacity, are defined operationally as the transmission rate per use of the channel, and also characterized by entropic expressions (see e.g. [39]). The only drawback of the entropic descriptions is the need of regularization, a process referring to entropic expression over many uses of the channel. This procedure is unavoidable in general for the quantum and private capacity [7, 13]. Furthermore, both one-shot capacities are potentially superadditive, i.e. the capacity of a combination of two channels may exceeds the sum of their individual capacities (see e.g. [19, 37, 36, 27]). Superadditivity is usually a quantum phenomenon due to entangled codings and decodings. In addition, it is desirable to have a quantum analog of the strong converse theorem from classical information theory. Quantum channels often leave open the question of whether there is a sharp trade off between the transmission rate and transmission accuracy, or there could exist an intermediate regime in which (where) errors are necessary but few.

Mathematically, a quantum channel is a completely positive and trace preserving map, which sends quantum states to quantum states. In recent years, quantum information theory has increasing interests in Rényi information measures, especially the sandwiched Rényi entropies introduced in [29, 43]. Some more implicit connection goes back to [9]. The

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sandwiched Rényi relative entropy and mutual information are used to prove the strong converse for entanglement-assisted communication [17], and to give an upper bound on the strong converse of classical communication [44] and quantum communication [38]. More recently, relative entropy of entanglement is shown to be a private strong converse rate via its sandwiched Rényi analogs [42].

The Rényi information measures are closely related to the Schatten p -norm of matrices

$$\|a\|_p = \text{tr}((a^*a)^{\frac{p}{2}})^{\frac{1}{p}}, \quad 1 \leq p \leq \infty,$$

and more generally, vector-valued noncommutative L_p -norms on tensor product spaces (multipartite systems). In this paper, we use complex interpolation to analyze quantum channels. The idea is related to our previous work [15], in which we proved an upper and lower bound (which differs up to a factor 2) on the quantum capacity of some nice classes of channels. The previous work was motivated by quantum group channels discovered in [23]. Here we simplify the algebraic assumptions and generalize our estimates to more channels and other capacities.

Let us recall that a quantum channel \mathcal{N} admits a Stinespring dilation:

$$\mathcal{N}(\rho) = \text{id} \otimes \text{tr}(U(\rho \otimes \phi)U^*)$$

where U is a unitary, tr the usual trace and ϕ an additional state. The particular case, $\phi = \frac{1}{d}$ being the completely mixed state on a d -dimensional Hilbert space is studied in [18], when disproving the quantum Birkhoff conjecture. More generally, we may obtain new channels by replacing tr by the normalized trace $\tau = \frac{\text{tr}}{d}$ and ϕ by a (normalized) density f . Indeed, we may consider

$$\mathcal{N}_f(\rho) = \text{id} \otimes \tau(U(\rho \otimes f)U^*),$$

as a modulated version of \mathcal{N}_1 for $f = 1$. Our main technique theorem identifies a condition on U that \mathcal{N}_f satisfies a “local comparison property”: for any positive operator $\sigma \in \text{Range}(\mathcal{N}_1)$ and ρ ,

$$\|\sigma^{-\frac{1}{2p'}} \mathcal{N}_1(\rho) \sigma^{-\frac{1}{2p'}}\|_p \leq \|\sigma^{-\frac{1}{2p'}} \mathcal{N}_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p \leq \|f\|_{p,\tau} \|\sigma^{-\frac{1}{2p'}} \mathcal{N}_1(\rho) \sigma^{-\frac{1}{2p'}}\|_p,$$

where $\|f\|_{p,\tau} = \tau(f^p)^{\frac{1}{p}}$ is the normalized L_p -norm. We apply this formula to the classical capacity C , the quantum capacity Q , and the private capacity P , and obtain

$$\begin{aligned} C(\mathcal{N}_1) &\leq C(\mathcal{N}_f) \leq C(\mathcal{N}_1) + \tau(f \log f), \quad Q(\mathcal{N}_1) \leq Q(\mathcal{N}_f) \leq Q(\mathcal{N}_1) + \tau(f \log f), \\ P(\mathcal{N}_1) &\leq P(\mathcal{N}_f) \leq P(\mathcal{N}_1) + \tau(f \log f). \end{aligned}$$

Similar estimates are proved for strong converse rates. We find nontrivial classes of examples for which the quantum capacity and the private capacity coincide with our upper bounds. In general, the estimates may not always be tight.

We organize this work as follows. Section 2 reviews the basics about channels, information measures and complex interpolation. In section 3, we recall the concept of TRO's from operator algebra theory and prove the “local comparison theorem”. For the class of quantum group channels similar result have been obtained in [15]. Section 4 is devoted to various applications on capacities, capacity regions and strong converse rates. Section 5 provides several examples that satisfy “local comparison theorem”.

2. PRELIMINARY

2.1. Channels and Stinespring spaces. We denote by $\mathbb{B}(H)$ the bounded operators on a Hilbert space H . The physical systems and their Hilbert spaces are indexed by capital letters as A, B, \dots . We restrict ourselves to finite dimensional Hilbert spaces. The standard n -dimensional Hilbert space is denoted by H_n and $n \times n$ matrix space is M_n . A *state* on H is given by a density operator $\rho \in \mathbb{B}(H)$, i.e. $\rho \geq 0$, $\text{tr}(\rho) = 1$, where “tr” represents the standard trace. We use superscripts to track the systems of multipartite state, i.e. for a state $\rho^{AB} \in \mathbb{B}(A \otimes B)$, $\rho^A = \text{id}_A \otimes \text{tr}_B(\rho^{AB})$ presents its reduced density matrix on A . We use 1_A (respectively 1_n) for the identity operator in $\mathbb{B}(A)$ (respectively M_n), and id_A (respectively id_n) for the identity map from $\mathbb{B}(A)$ (respectively M_n) to itself.

A linear map $\mathcal{N} : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$ is a quantum channel if it is completely positive and trace preserving, i.e. all its amplification $\text{id}_n \otimes \mathcal{N} : \mathbb{B}(A'^{\otimes n}) \rightarrow \mathbb{B}(B^{\otimes n})$ send states to states. An equivalent definition is given by the *Stinespring dilation*: there exists a Hilbert space E and an isometry $V : A' \rightarrow B \otimes E$ with $V^*V = \text{id}_{A'}$ such that

$$\mathcal{N}(\rho) = \text{id}_B \otimes \text{tr}_E(V\rho V^*),$$

where tr_E stands for the standard trace on $\mathbb{B}(E)$. The complementary channel of \mathcal{N} is given by

$$\mathcal{N}^E(\rho) = \text{tr}_B \otimes \text{id}_E(V\rho V^*). \quad (2.1)$$

The dilation (2.1) is not unique, but different ones are related by a partial isometry between the environment systems. The image of the isometry $\text{Im}(V)$, as a subspace of the tensor product Hilbert space $B \otimes E$, is called the *Stinespring space* of \mathcal{N} . A channel \mathcal{N} is completely decided by its Stinespring isometry V , and much information about the channel is encoded in its Stinespring space. In particular, Stinespring space was used in disproving the additivity conjecture for the minimal entropy ([2]).

Given an orthonormal basis $\{|e_i\rangle\}$ of H_E and its dual basis $\{\langle e_i|\}$ in H_E^* , one can identify the tensor Hilbert space $B \otimes E$ with the operators $\mathbb{B}(E, B)$ as follows,

$$|h\rangle = \sum_i |h_i\rangle \otimes |e_i\rangle \rightarrow h = \sum_i |h_i\rangle \otimes \langle e_i|, \quad |h_i\rangle \in H_B.$$

This map depends on the choice of the basis $\{|e_i\rangle\}$ but is unique up to a unitary equivalence. It plays a role as the partial trace on pure bipartite states,

$$id_B \otimes tr_E(|h\rangle\langle k|) = hk^* , \quad tr_B \otimes id_E(|h\rangle\langle k|) = k^*h . \quad (2.2)$$

Here the second equality is valid up to a unitary equivalence. Throughout this paper we will always use “bracket” notations for vectors/dual vectors. The Stinespring space $X = Im(V)$ then becomes an operator subspace of $\mathbb{B}(E, B)$. For a vector $|h\rangle \in A'$, we denote by $\widehat{V}h$ the corresponding operator to the vector $V|h\rangle \in B \otimes E$. With this notation, we can rewrite the channel and its complementary channel as

$$\mathcal{N}(|h\rangle\langle k|) = \widehat{V}h\widehat{V}k^* , \quad \mathcal{N}^E(|h\rangle\langle k|) = \widehat{V}k^*\widehat{V}h .$$

Again, the second equality holds up to a unitary equivalence.

2.2. Information measures and noncommutative L_p -norms. As we will use L_p -norms, p always denote a real number in $[1, \infty]$ and p' is its conjugate $1/p + 1/p' = 1$. The Schatten p class operator on H (respectively H_n) is denoted as $S_p(H)$ (S_p^n). The norms are written as

$$\|a\|_{S_p(H)} = tr((a^*a)^{\frac{p}{2}})^{\frac{1}{p}} ,$$

and sometimes also shorten as $\|\cdot\|_p$ if the underlying space is clear.

Given a state ρ , its von Neumann entropy $H(\rho)$ and p -Rényi entropy $H_p(\rho)$ are related to Schatten p -norms as follows,

$$H(\rho) = -tr(\rho \log \rho) = \lim_{p \rightarrow 1} H_p(\rho) , \quad H_p(\rho) = -\frac{p}{p-1} \log \|\rho\|_p .$$

For information theoretic purpose, the logarithm “log” (and the exponential “exp”) will be of base 2, which differs with the derivatives of usual L_p -norms with a scalar $\ln 2$. We will ignore this constant for simplicity. For two positive operators ρ, σ on H , the relative entropy $D(\rho||\sigma)$ is defined as,

$$D(\rho||\sigma) = \begin{cases} tr(\rho \log \rho - \rho \log \sigma) & \text{if } \text{supp}(\rho) \subset \text{supp}(\sigma) \\ \infty & \text{else} \end{cases} .$$

The sandwiched relative p -Rényi entropy $D_p(\rho||\sigma)$ was introduced in [29, 43]. For $1 \leq p \leq \infty$, it can be written with Schatten p -norms as follows,

$$D_p(\rho||\sigma) = \begin{cases} p' \log \|\sigma^{-\frac{1}{2p'}} \rho \sigma^{-\frac{1}{2p'}}\|_p & \text{if } \text{supp}(\rho) \subset \text{supp}(\sigma) \\ \infty & \text{else} \end{cases} , \quad D(\rho||\sigma) = \lim_{p \rightarrow 1} D_p(\rho||\sigma) .$$

For a bipartite state ρ^{AB} , the conditional entropy $H(A|B)_\rho$ and its sandwiched p -Rényi analog $H_p(A|B)$ can be defined via the relative entropy [29],

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho = -\inf_{\sigma^B} D(\rho^{AB}||1_A \otimes \sigma^B) ,$$

$$H_p(A|B)_\rho = -\inf_{\sigma^B} D_p(\rho^{AB} \| 1_A \otimes \sigma^B),$$

where the infimums run over all states σ^B on B . The latter one connects to the vector-valued noncommutative L_p -spaces introduced by Pisier (see [32]). Indeed, let $1 \leq p, q \leq \infty$ and fix $1/r = |1/p - 1/q|$. For a bipartite operator $\rho \in \mathbb{B}(A \otimes B)$, the $S_q(B, S_p(A))$ norms are given as follows: for $p \leq q$, $1/q + 1/r = 1/p$

$$\|\rho\|_{S_q(A, S_p(B))} = \sup_{\|a\|_{S_{2r}(A)} \|b\|_{S_{2r}(A)} \leq 1} \|(a \otimes 1_B) \rho (b \otimes 1_B)\|_{S_p(A \otimes B)}, \quad (2.3)$$

and for $p \geq q$, $1/p + 1/r = 1/q$,

$$\|\rho\|_{S_q(A, S_p(B))} = \inf_{\rho=(a \otimes 1_B) \eta (b \otimes 1_B)} \|a\|_{S_{2r}(H_A)} \|\eta\|_{S_p(H_A \otimes H_B)} \|b\|_{S_{2r}(H_A)}. \quad (2.4)$$

When ρ is positive, it is sufficient to choose $a = b \geq 0$ in (2.3) and (2.4), and then the $(1, p)$ norm connects to the sandwiched Rényi conditional entropy as follows,

$$-\frac{p}{p-1} \log \|\rho\|_{S_1(B, S_p(A))} = H_p(A|B)_\rho.$$

All the limits all above can be interpreted as derivatives at $p = 1$,

$$\begin{aligned} H(\rho) &= -\frac{d}{dp} \Big|_{p=1} \|\rho\|_p, \quad H(A|B)_\rho = -\frac{d}{dp} \Big|_{p=1} \|\rho\|_{S_p(H_B, S_p(H_A))}, \\ D(\rho|\sigma) &= \frac{d}{dp} \Big|_{p=1} \|\sigma^{-\frac{1}{2p'}} \rho \sigma^{-\frac{1}{2p'}}\|_p. \end{aligned}$$

This observation enable us to translate norm estimates to entropic inequalities.

2.3. Complex interpolation. Two Banach spaces X_0 and X_1 are compatible if there exists a Hausdorff topological vector space X such that $X_0, X_1 \subset X$ as subspaces. The sum space $X_0 + X_1$ is a Banach space

$$X_0 + X_1 := \{x \in X \mid x = x_0 + x_1 \text{ for some } x_0 \in X_0, x_1 \in X_1\},$$

equipped with the norm

$$\|x\|_{X_0+X_1} = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + \|x_1\|_{X_1}).$$

Let $S = \{z \mid 0 \leq \operatorname{Re}(z) \leq 1\}$ be the vertical strip of unit width on the complex plane, and let $S_0 = \{z \mid 0 < \operatorname{Re}(z) < 1\}$ be its open interior. We denote by $\mathcal{F}(X_0, X_1)$ the space of all functions $f : S \rightarrow X_0 + X_1$, which are bounded and continuous on S and analytic on S_0 , and moreover

$$\{f(it) \mid t \in \mathbb{R}\} \subset X_0, \quad \{f(1+it) \mid t \in \mathbb{R}\} \subset X_1.$$

$\mathcal{F}(X_0, X_1)$ is again a Banach space with the norm

$$\|f\|_{\mathcal{F}} = \max\left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{X_1} \right\}.$$

The complex interpolation space $(X_0, X_1)_\theta$, for $0 < \theta < 1$, is the quotient space of $\mathcal{F}(X_0, X_1)$ given as follows,

$$(X_0, X_1)_\theta = \{ x \in X_0 + X_1 \mid x = f(\theta), f \in \mathcal{F}(X_0, X_1) \}.$$

The quotient norm is defined as

$$\|x\|_\theta = \inf \{ \|f\|_{\mathcal{F}} \mid f(\theta) = x \}.$$

For example, the Schatten- p class is the interpolation space of bound operator and trace class

$$S_p(H) = (B(H), S_1(H))_{\frac{1}{p}}.$$

This generalizes to vector-valued noncommutative L_p space $S_p(A, S_q(B))$ (see [32]). In particular, for any $1 \leq q \leq \infty$ one has the relation

$$S_p(A, S_q(B)) = [S_\infty(A, S_q(B)), S_1(A, S_q(B))]_{\frac{1}{p}}.$$

The following Stein's interpolation theorem (cf. [4]) is a key tool in our analysis.

Theorem 2.1. *Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of Banach spaces. Let $\{T_z \mid z \in S\} \subset \mathbb{B}(X_0 + X_1, Y_0 + Y_1)$ be a bounded analytic family of maps such that*

$$\{T_{it} \mid t \in \mathbb{R}\} \subset \mathbb{B}(X_0, Y_0), \quad \{T_{1+it} \mid t \in \mathbb{R}\} \subset \mathbb{B}(X_1, Y_1).$$

Suppose $\Lambda_0 = \sup_t \|T_{it}\|_{\mathbb{B}(X_0, Y_0)}$ and $\Lambda_1 = \sup_t \|T_{1+it}\|_{\mathbb{B}(X_1, Y_1)}$ are both finite, then for $0 < \theta < 1$, T_θ is a bounded linear map from $(X_0, X_1)_\theta$ to $(Y_0, Y_1)_\theta$ and

$$\|T_\theta\|_{\mathbb{B}((X_0, X_1)_\theta, (Y_0, Y_1)_\theta)} \leq \Lambda_0^{1-\theta} \Lambda_1^\theta.$$

In particular, when T is a constant map, the above theorem implies

$$\|T\|_{\mathbb{B}((X_0, X_1)_\theta, (Y_0, Y_1)_\theta)} \leq \|T\|_{\mathbb{B}(X_0, Y_0)}^{1-\theta} \|T\|_{\mathbb{B}(X_1, Y_1)}^\theta. \quad (2.5)$$

3. TRO CHANNELS

Let us recall that a *ternary ring of operators* (TRO) X between Hilbert spaces H and K is a closed subspace of $\mathbb{B}(H, K)$ closed under the triple product

$$x, y, z \in X \Rightarrow xy^*z \in X.$$

They were first introduced by Hestenes [20], and pursued by many others (see e.g. [25, 31]). A TRO X is a corner of its linking C^* -algebra $\mathcal{L}(X)$ introduced in [5],

$$\mathcal{L}(X) = \text{span} \left\{ \begin{bmatrix} xy^* & z \\ w^* & v^*u \end{bmatrix} \mid x, y, z, u, v, w \in X \right\} = \begin{bmatrix} B(X) & X \\ X^* & E(X) \end{bmatrix}.$$

The two diagonal blocks are C^* -algebras,

$$B(X) = \text{span} \{ xy^* \mid x, y \in X \} \subset \mathbb{B}(K), \quad E(X) = \text{span} \{ x^*y \mid x, y \in X \} \subset \mathbb{B}(H).$$

We call $B(X)$ the left C^* -algebra of X and $E(X)$ the right C^* -algebra. They play an important role in the study of TROs (see again e.g. [25]). In particular, X is a natural $B(X) - E(X)$ bimodule

$$B(X)X = X, \quad XE(X) = X.$$

It can be shown that in finite dimensions, a TRO is given by a direct sum of rectangular matrices with multiplicity. Namely, for a TRO space X in $n \times m$ matrices $M_{n,m}$, we may assume that in some orthonormal bases $X = \oplus_i M_{n_i, m_i} \otimes \mathbb{C}1_{l_i}$, where l_i is the multiplicity of i th diagonal block and $\sum_i n_i l_i = n$, $\sum_i m_i l_i = m$ match the dimensions. In this situation,

$$B(X) = \oplus_i M_{n_i} \otimes \mathbb{C}1_{l_i} \subset M_n, \quad E(X) = \oplus_i M_{m_i} \otimes \mathbb{C}1_{l_i} \subset M_m.$$

In most of our discussions, the multiplicities l_i are irrelevant and we may often use the notation $X \cong \oplus_i M_{n_i, m_i}$ for simplicity.

Let X_p denote the closure of intersection $X \cap S_p(H, K)$ in the Schatten p class. TROs and their corresponding subspaces in $S_p(H, K)$ are completely 1-complemented for $1 \leq p \leq \infty$ [11, 30]. That is, there exists a projection map \mathcal{P} from $\mathbb{B}(H, K)$ (respectively, $S_p(H, K)$) onto X (respectively, X_p) such that $id_n \otimes \mathcal{P}$ is contractive for every n . A direct consequence is that X_p are interpolation spaces of X_1 and X ,

$$X_p = (X_\infty, X_1)_{\frac{1}{p}}.$$

More generally, one has the following simple application of Kosaki-type interpolation [26].

Theorem 3.1. *Let $X \subset \mathbb{B}(H, K)$ be a TRO. For a positive operator $\sigma \in B(X)$ and $1 \leq p \leq \infty$, define $X_{p,\sigma}$ as the space X equipped with the following norms,*

$$\|x\|_{p,\sigma} := \|\sigma^{\frac{1}{p}} x\|_p.$$

Then

$$[X_\infty, X_{1,\sigma}]_{\frac{1}{p}} = X_{p,\sigma}.$$

Proof. Let us first assume that σ is invertible. For $x \in X$ such that $\|\sigma^{\frac{1}{p}} x\|_p = 1$, we consider the polar decomposition $\sigma^{\frac{1}{p}} x = v|\sigma^{\frac{1}{p}} x| := vy$, where $v \in X$ is a partial isometry and $y \in B(X)$. Then we define the analytic function x from the strip $S = \{z \mid 0 \leq \operatorname{Re}(z) \leq 1\}$ to X as follows,

$$x(z) = \sigma^{-z} v y^{pz}, \quad x(1/p) = \sigma^{-\frac{1}{p}} v y = x.$$

Note that

$$\|x(it)\|_\infty = \|\sigma^{-it} v y^{itp}\|_\infty \leq 1, \quad \|x(1+it)\|_{1,\sigma} = \|\sigma \sigma^{-1-it} v y^{p(1+it)}\|_1 = \|v y^p\|_1 \leq \|v y\|_p^p \leq 1.$$

Therefore $\|x\|_{[X_\infty, X_{1,\sigma}]^{\frac{1}{p}}} \leq \|\sigma^{\frac{1}{p}}x\|_p = 1$. On the other hand, suppose that we have an analytic function $x : S \rightarrow X$ such that

$$\sup_t \{ \|x(it)\|_\infty, \|\sigma x(1+it)\|_1 \} \leq 1.$$

Recall that $1/p' + 1/p = 1$. For any $\|a\|_{S_{p'}(K,H)} \leq 1$, we claim

$$\text{tr}(\sigma^{1/p}xa) \leq 1.$$

Indeed, consider the analytic function $h(z) = \text{tr}(\sigma^z x(z)a(z))$ where $a(z) = w|a|^{p'(1-z)}$. On the boundary of the strip S ,

$$|h(it)| \leq \|x(it)\|_\infty \|a(it)\|_1 \leq 1, \quad |h(1+it)| \leq \|\sigma x(1+it)\|_1 \|a(1+it)\|_\infty \leq 1.$$

By the maximum principle, we obtain that $|h(1/p)| = |\text{tr}(\sigma^{\frac{1}{p}}xy)| \leq 1$, which proves the claim. For noninvertible σ , one can repeat the argument for $\tilde{\sigma} = \sigma + \delta 1$ with $\delta > 0$ and let δ go to 0. \blacksquare

Remark 3.2. The above interpolation relation can be generalized to two-sided densities. Let $0 \leq \theta \leq 1$. Given $\sigma \in B(X)$, $\rho \in E(X)$, one can define $X_{p,\theta}$ as the corresponding space equipped with the norm,

$$\|x\|_{p,\theta,\sigma,\rho} = \|\sigma^{\frac{\theta}{p}}x\rho^{\frac{1-\theta}{p}}\|_p.$$

These L_p -spaces also interpolate [24],

$$[X_\infty, X_{1,\theta,\sigma,\rho}]^{\frac{1}{p}} = X_{p,\theta,\sigma,\rho}.$$

Let us denote by τ the normalized trace $\frac{\text{tr}}{\dim H}$ on $\mathbb{B}(H)$ and $\|\cdot\|_{p,\tau}$ be the corresponding normalized L_p -norm. We use the notation with subscripts τ_A to specify the normalized trace on $\mathbb{B}(A)$. A positive operator f is called a normalized density if $\tau(f) = 1$. Given a C^* -subalgebra $M \subset \mathbb{B}(H)$, the *conditional expectation* \mathcal{E}_M is the unique completely positive, trace preserving and unital map from $\mathbb{B}(H)$ onto M ($M + \mathbb{C}1$ if nonunital) such that

$$\tau(\mathcal{E}_M(x)y) = \tau(xy) \quad \text{for } x \in \mathbb{B}(H), y \in M. \quad (3.1)$$

We say that two C^* -subalgebras M and N are independent if $\mathcal{E}_M \circ \mathcal{E}_N = \mathcal{E}_N \circ \mathcal{E}_M = \tau$. Equivalently, this means

$$\tau(xy) = \tau(x)\tau(y) \quad \text{or} \quad \text{tr}(xy) = \tau(x)\text{tr}(y) = \text{tr}(x)\tau(y), \quad \forall x \in M, y \in N.$$

Now we are ready to define our TRO channels. Let $X \subset \mathbb{B}(E, B)$ be a TRO. Equipped with the Hilbert-Schmidt norm, X_2 is a Hilbert subspace of $B \otimes E$ by assigning a vector $|x\rangle$ to each operator $x \in X$ and the inner product is given by $\langle x|y\rangle = \text{tr}(x^*y)$. Let N be

a C^* -algebra independent of the right algebra $E(X)$. Then for any $f \in N$, we can define the following map

$$\mathcal{N}_f : \mathbb{B}(X_2) \rightarrow \mathbb{B}(B), \mathcal{N}_f(|x\rangle\langle y|) = xfy^*.$$

Proposition 3.3. *Let \mathcal{N}_f be defined as above. Then*

- i) *for any $f \in N$, $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \tau(f)\mathcal{N}_1$;*
- ii) *for a normalized density f in N , \mathcal{N}_f is a quantum channel and its Stinespring isometry is given by*

$$V_f|x\rangle = |x\sqrt{f}\rangle.$$

Proof. Let $a \in B(X)$ and $|x\rangle, |y\rangle \in X_2$. By independence, we have that

$$\begin{aligned} \tau_B(a\mathcal{N}_f(|x\rangle\langle y|)) &= \frac{1}{\dim B} \text{tr}_B(axfy^*) = \frac{1}{\dim B} \text{tr}_E(y^*axf) \\ &= \tau_E(f) \text{tr}_E(y^*ax) = \frac{1}{\dim B} \tau_E(f) \text{tr}_B(axy^*) = \tau_E(f) \tau_B(a\mathcal{N}_1(|x\rangle\langle y|)). \end{aligned}$$

Thus,

$$\mathcal{E}_{B(X)} \circ \mathcal{N}_f(|x\rangle\langle y|) = \tau(f)\mathcal{N}_1(|x\rangle\langle y|)$$

holds for any rank one matrix $|x\rangle\langle y|$. By linearity we prove i). For ii), note that

$$\mathcal{N}_f(|x\rangle\langle y|) = xfy^* = id_B \otimes \text{tr}_E(|x\sqrt{f}\rangle\langle y\sqrt{f}|).$$

Then it is sufficient to verify that V_f is an isometry given $\tau(f) = 1$. Indeed, we have

$$\| |x\sqrt{f}\rangle \|^2 = \text{tr}_B(xfx^*) = \text{tr}_E(x^*xf) = \tau_E(f) \text{tr}_E(x^*x) = \| |x\rangle \|^2.$$

In the second last step we use the assumptions that f is independent of $E(X)$. ■

Definition 1. *Let X be a TRO in $\mathbb{B}(B, E)$. We say a normalized density $f \in B(E)$ is a symbol for X if $C^*(f)$ is independent of $E(X)$. For each symbol f , we call the channel*

$$\mathcal{N}_f(|x\rangle\langle y|) = xfy^*$$

a X -TRO channel with symbol f . For $f = 1$, we call \mathcal{N}_1 the X -TRO channel and simply write \mathcal{N} .

This definition generalizes the “vN-channels” introduced in our previous work [15]. There we used a model based on an abstraction of quantum group structure. See Section 5.1 for detailed discussion about the relations between the two setups.

Our main technical theorem is the following “local comparison theorem”.

Theorem 3.4. *Let \mathcal{N}_f be a X -TRO channel with symbol f . Then for any positive operators $\sigma \in B(X)$ and ρ ,*

$$\| \sigma^{-\frac{1}{2p'}} \mathcal{N}(\rho) \sigma^{-\frac{1}{2p'}} \|_p \leq \| \sigma^{-\frac{1}{2p'}} \mathcal{N}_f(\rho) \sigma^{-\frac{1}{2p'}} \|_p \leq \| f \|_{p,\tau} \| \sigma^{-\frac{1}{2p'}} \mathcal{N}(\rho) \sigma^{-\frac{1}{2p'}} \|_p.$$

Proof. The conditional expectation $\mathcal{E}_{B(X)}$ by definition is quantum channel. From Proposition 3.3, we know that $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \mathcal{N}$ and $\mathcal{E}_{B(X)}(\sigma) = \sigma$ for $\sigma \in B(X)$. Then the first inequality of (3.4) follows from the data processing inequality of Rényi sandwiched relative entropy D_p (see e.g. [29]),

$$D_p(\mathcal{N}_f(\rho) || \sigma) \geq D_p(\mathcal{E}_{B(X)} \circ \mathcal{N}_f(\rho) || \mathcal{E}_{B(X)}(\sigma)) = D_p(\mathcal{N}(\rho) || \sigma).$$

Let σ^{-1} denote the inverse of σ on its support. Write $\rho = \eta\eta^*$ with $\eta \in \mathbb{B}(X_2, A)$ for some Hilbert space A , then

$$\|\sigma^{-\frac{1}{2p'}} \mathcal{N}_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p = \|\sigma^{-\frac{1}{2p'}} \widehat{V_f \eta}\|_{S_{2p}(A \otimes E, B)}.$$

Here we used the notation $\widehat{V\eta}$ to represent corresponding operator in $\mathbb{B}(A \otimes E, B)$. Note that $\widehat{V_f \eta} = \widehat{V\eta}(1_A \otimes f^{\frac{1}{2}})$, then for the second inequality, it is sufficient to show that

$$\|\sigma^{-\frac{1}{2p'}} \widehat{V\eta}(1_A \otimes f^{\frac{1}{2}})\|_{S_{2p}(A \otimes E, B)} \leq \|f^{\frac{1}{2}}\|_{2p, \tau} \|\sigma^{-\frac{1}{2p'}} \widehat{V\eta}\|_{S_{2p}(A \otimes E, B)}.$$

Assume that $\|f^{\frac{1}{2}}\|_{2p, \tau} \|\sigma^{-\frac{1}{2p'}} \widehat{V\eta}\|_{2p} < 1$, we have $\|\sigma^{\frac{1}{2p}} \xi\|_{2p} < 1$ where $\xi = \sigma^{-\frac{1}{2}} \widehat{V\eta} \in X \otimes \mathbb{B}(A, \mathbb{C})$. Note that $\tilde{X} \equiv X \otimes \mathbb{B}(A, \mathbb{C}) \subset \mathbb{B}(A \otimes E, B)$ is again a TRO channel with $B(\tilde{X}) = B(X)$, then by Proposition 3.1 and the reiteration theorem (see e.g. [4]), we know that

$$\tilde{X}_{\sigma, 2p} = [\tilde{X}_\infty, \tilde{X}_{\sigma, 2}]_{\frac{1}{p}}.$$

Therefore there exists an analytic function $\xi : S = \{z | 0 \leq \operatorname{Re}(z) \leq 1\} \rightarrow \tilde{X}$ such that $\xi(1/p) = \xi$ and moreover

$$\|\xi(it)\|_\infty < 1, \quad \|\sigma^{\frac{1}{2}} \xi(1+it)\|_2 < 1.$$

Given this, we define another analytic function $T(z) = \sigma^{\frac{z}{2}} \xi(z)(1_A \otimes a^{pz})$, where $a = \frac{f^{\frac{1}{2}}}{\|f^{\frac{1}{2}}\|_{2p, \tau}}$. Observe that

$$\begin{aligned} \|T(it)\|_\infty &= \|\sigma^{\frac{it}{2}} \xi(it) 1_A \otimes a^{\frac{itp}{2}}\|_\infty = \|\xi(it)\|_\infty < 1, \\ \|T(1+it)\|_2 &= \|\sigma^{\frac{1+it}{2}} \xi(1+it) (1 \otimes a^{\frac{p(1+it)}{2}})\|_2 = \|\sigma^{\frac{1}{2}} \xi(1+it) (1 \otimes a^{\frac{p}{2}})\|_2 \\ &= \|\sigma^{\frac{1}{2}} \xi(1+it)\|_2 \|a^{\frac{p}{2}}\|_2 < 1. \end{aligned}$$

The second last equality follows from the fact $a \in C^*(f)$ independent with $E(X)$. The last equality use the fact $\|a^{\frac{p}{2}}\|_2 = 1$. By Stein's interpolation theorem (3.1), we obtain

$$\|T(\frac{1}{p})\|_{2p} = \|\sigma^{\frac{1}{p}} \xi(1/p) 1_A \otimes a^{\frac{1}{2}}\|_{2p} = \frac{1}{\|\sqrt{f}\|_{2p, \tau}} \|\sigma^{-\frac{1}{2p'}} \widehat{V_f \eta}\|_{2p} \leq 1,$$

which completes the proof. ■

The above result is a local property which applies for every ρ . We can naturally consider the restrictions of TRO channels on subspaces. Recall that we use the notation $\widehat{V}h$ for the operator in $\mathbb{B}(E, B)$ corresponding to the vector $V|h\rangle \in B \otimes E$.

Definition 2. Let $\mathcal{N} : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$ be a quantum channel with Stinespring isometry $V_{\mathcal{N}} : A' \rightarrow B \otimes E$. We call a normalized density $f \in \mathbb{B}(E)$ an admissible symbol for \mathcal{N} if $C^*(f)$ is independent of the C^* -algebra generated by $\{\widehat{V}h^* \widehat{V}k | h, k \in A'\}$. For each admissible symbol f , we define the modulated channel \mathcal{N}_f as follows,

$$\mathcal{N}_f(|h\rangle\langle k|) = \widehat{V}h f \widehat{V}k^*, \quad |h\rangle, |k\rangle \in A'. \quad (3.2)$$

Remark 3.5. a) Let $Y = V(A') \subset \mathbb{B}(E, B)$ be a subspace, $E(Y)$ be the right C^* -algebra generated by $\{x^*y | x, y \in Y\}$ and $B(Y)$ be the left C^* -algebra generated by $\{yx^* | y, x \in Y\}$. Then $X = YE(Y) = B(Y)Y$ is a TRO. Therefore every admissible symbol gives rise to a TRO channel $\mathcal{M}_f(|x\rangle\langle y|) = xfy^*$ and the modulation \mathcal{N}_f is the restriction of \mathcal{M}_f on $\mathbb{B}(A')$, a corner in $\mathbb{B}(X_2)$. Indeed, the isometry $V : A' \rightarrow X_2$ is the inclusion when $X_2 \subset B \otimes E$ is equipped with the Hilbert Schmidt norm.

b) Using this terminology every channel is a restriction of a TRO channel with a trivial symbol 1. However, the smallest TRO obtained from the minimal Stinespring dilation may produce a large algebra $B(V(A'))$, and hence our estimates may not be effective. Instead, for a given channel Φ it is better to first identify a TRO channel with small left algebra $B(X)$ and rewrite $\Phi = \mathcal{N}_f$ for a suitable admissible symbol.

Since all the local properties automatically generalizes to the restrictions of TRO channels, we obtain the following corollary.

Corollary 3.6. Let \mathcal{N} be a quantum channel. Assume that f is an admissible symbol for \mathcal{N} with respect to a TRO X , then

- i) $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \mathcal{N}$;
- ii) for any positive operators $\sigma \in B(X)$ and ρ ,

$$\|\sigma^{-\frac{1}{2p'}} \mathcal{N}(\rho) \sigma^{-\frac{1}{2p'}}\|_p \leq \|\sigma^{-\frac{1}{2p'}} \mathcal{N}_f(\rho) \sigma^{-\frac{1}{2p'}}\|_p \leq \|f\|_{p,\tau} \|\sigma^{-\frac{1}{2p'}} \mathcal{N}(\rho) \sigma^{-\frac{1}{2p'}}\|_p.$$

The definition of a symbol is compatible with tensor products.

Lemma 3.7. Let f be an admissible symbol for \mathcal{N} and g be an admissible symbol for \mathcal{M} . Then $f \otimes g$ is an admissible symbol for $\mathcal{N} \otimes \mathcal{M}$. Moreover, we have $(\mathcal{N} \otimes \mathcal{M})_{f \otimes g} = \mathcal{N}_f \otimes \mathcal{M}_g$. In particular, for any channel \mathcal{M} the identity 1 is always an admissible symbol and $(\mathcal{N} \otimes \mathcal{M})_{f \otimes 1} = \mathcal{N}_f \otimes \mathcal{M}$.

Proof. Let $X^{\mathcal{N}}$ be the Stinespring spaces of \mathcal{N} and $X^{\mathcal{M}}$ be the Stinespring spaces of \mathcal{M} . Since f and g are admissible symbols for \mathcal{N} and \mathcal{M} respectively, there exists TROs

$X \subset \mathbb{B}(E, B)$ containing $X^{\mathcal{N}}$ and $X_1 \subset \mathbb{B}(E_1, B_1)$ containing $X^{\mathcal{M}}$ such that $C^*(f)$ is independent of $E(X)$ and $C^*(g)$ is independent of $E(X_1)$. Then

$$X \otimes X_1 \subset \mathbb{B}(E, B) \otimes \mathbb{B}(E_1, B_1) \cong \mathbb{B}(E \otimes E_1, B \otimes B_1)$$

is a TRO containing $X^{\mathcal{N} \otimes \mathcal{M}} = X^{\mathcal{N}} \otimes X^{\mathcal{M}}$. It is easy to see that $C^*(f \otimes g) = C^*(f) \otimes C^*(g)$ is still independent of $E(X \otimes X_1) = E(X) \otimes E(X_1)$. Moreover, $f \otimes g$ is again a normalized density hence an admissible symbol for $\mathcal{N} \otimes \mathcal{M}$. Let V be Stinespring isometry of \mathcal{N} and V_1 of \mathcal{M} . For $|h\rangle, |k\rangle \in A', |h_1\rangle, |k_1\rangle \in A'_1$,

$$\begin{aligned} (\mathcal{N} \otimes \mathcal{M})_{f \otimes g}(|h\rangle\langle k| \otimes |h_1\rangle\langle k_1|) &= (V \otimes \widehat{V_1})(\widehat{h \otimes h_1})(f \otimes g)(V \otimes \widehat{V_1})(\widehat{k \otimes k_1})^* \\ &= (\widehat{Vh}f\widehat{V}k^*) \otimes (\widehat{V_1h_1}g\widehat{V_1}k_1^*) = N_f(|h\rangle\langle k|) \otimes M_g(|h_1\rangle\langle k_1|). \end{aligned}$$

That finishes the proof. ■

Corollary 3.8. *Let \mathcal{N} be a channel and f be an admissible symbol for \mathcal{N} . Let A be an arbitrary Hilbert space and ρ in $\mathbb{B}(A' \otimes A)$ positive operator. Then*

- i) $\|\mathcal{N} \otimes id_A(\rho)\|_p \leq \|\mathcal{N}_f \otimes id_A(\rho)\|_p \leq \|f\|_{p,\tau} \|\mathcal{N} \otimes id_A(\rho)\|_p$;
- ii) $\|\mathcal{N} \otimes id_A(\rho)\|_{S_1(B, S_p(A))} \leq \|\mathcal{N}_f \otimes id_A(\rho)\|_{S_1(B, S_p(A))} \leq \|f\|_{p,\tau} \|\mathcal{N} \otimes id_A(\rho)\|_{S_1(B, S_p(A))}$.

Proof. By Lemma 3.7, $f \otimes 1$ is an admissible symbol for $\mathcal{N} \otimes id_A$ and $(\mathcal{N} \otimes id_A)_{f \otimes 1} = \mathcal{N}_f \otimes id_A$. Thus i) follows from Theorem 3.6 by choosing $\sigma = 1_{BA}$. For ii), we assume that ρ is a state ($tr(\rho) = 1$). Denote $\omega_f = \mathcal{N}_f \otimes id_A(\rho)$ and $\omega = \mathcal{N} \otimes id_A(\rho)$ as the output states. Let X be a TRO containing the Stinespring space $X^{\mathcal{N}}$ and independent with $C^*(f)$. Recall that we have the factorization $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \mathcal{N}$ from Proposition 3.3. Then we apply the data processing inequality and obtain

$$\|\omega_f\|_{S_1(B, S_p(A))} = \exp\left(-\frac{1}{p'} H_p(A|B)_{\omega_f}\right) \geq \exp\left(-\frac{1}{p'} H_p(A|B)_{\omega_1}\right) = \|\omega\|_{S_1(B, S_p(A))},$$

which proves the first half of ii). For the second, applying Theorem 3.4,

$$\begin{aligned} \|\omega_f\|_{S_1(B, S_p(A))} &= \inf_{\sigma_B} \|\sigma^{-\frac{1}{2p'}} \omega_f \sigma^{-\frac{1}{2p'}}\|_p \leq \inf_{\sigma_B \in B(X)} \|\sigma^{-\frac{1}{2p'}} \omega_f \sigma^{-\frac{1}{2p'}}\|_p \\ &\leq \|f\|_{p,\tau} \inf_{\sigma_B \in B(X)} \|\sigma^{-\frac{1}{2p'}} \omega \sigma^{-\frac{1}{2p'}}\|_p. \end{aligned}$$

On the other hand, note that $\mathcal{E}_{B(X)} \otimes id_A(\omega) = \omega$, then

$$\begin{aligned} \|\omega\|_{S_1(B, S_p(A))} &= \inf_{\sigma_B} \|(\sigma^{-\frac{1}{2p'}} \otimes 1_A) \omega (\sigma^{-\frac{1}{2p'}} \otimes 1_A)\|_p = \inf_{\sigma_B} \exp\left(\frac{1}{p'} D_p(\omega \| \sigma_B \otimes 1_A)\right) \\ &\geq \inf_{\sigma_B} \exp\left(\frac{1}{p'} D_p(\omega \| \mathcal{E}_{B(X)}(\sigma_B) \otimes 1_A)\right) \end{aligned}$$

$$\begin{aligned}
&= \inf_{\sigma_B \in B(X)} \exp\left(\frac{1}{p'} D_p(\omega \| \sigma_B \otimes 1_A)\right) \\
&\geq \inf_{\sigma_B} \|(\sigma^{-\frac{1}{2p'}} \otimes 1_A) \omega (\sigma^{-\frac{1}{2p'}} \otimes 1_A)\|_p .
\end{aligned}$$

Hence we obtain

$$\|\omega\|_{S_1(B, S_p(A))} = \inf_{\sigma_B \in B(X)} \|(\sigma^{-\frac{1}{2p'}} \otimes 1_A) \omega (\sigma^{-\frac{1}{2p'}} \otimes 1_A)\|_p ,$$

which completes proof. ■

4. APPLICATIONS

4.1. Entropic inequalities. Recall that for a bipartite state ρ^{AB} , the coherent information $I_c(A)B)_\rho$ and mutual information $I(A : B)_\rho$ are defined as follows,

$$I_c(A)B)_\rho = H(\rho^{AB}) - H(\rho^B) , \quad I(A : B)_\rho = H(\rho^A) + H(\rho^B) - H(\rho^{AB}) .$$

The first application of the local comparison property is about entropic inequalities.

Corollary 4.1. *Let $\mathcal{N}_f : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$ be a channel and f be an admissible symbol for \mathcal{N} . Denote $\omega_f^{AB} = id_A \otimes \mathcal{N}_f(\rho^{AA'})$ and $\omega^{AB} = id_A \otimes \mathcal{N}(\rho^{AA'})$ respectively for a bipartite state $\rho^{AA'}$. Then the following inequalities hold:*

- i) $H(AB)_\omega - \tau(f \log f) \leq H(AB)_{\omega_f} \leq H(AB)_\omega$;
- ii) $I_c(A)B)_\omega \leq I_c(A)B)_{\omega_f} \leq I_c(A)B)_\omega + \tau(f \log f)$;
- iii) $I(A; B)_\omega \leq I(A; B)_{\omega_f} \leq I(A; B)_\omega + \tau(f \log f)$.

Proof. i) and ii) follows from Corollary 3.8 by taking derivatives at $p = 1$. iii) is a consequence of ii) because $I(A : B) = H(A) + I_c(A)B)$ and $\omega_f^A = \omega^A$. ■

Remark 4.2. The term $\tau(f \log f)$ corresponds to the entropy for the normalized trace τ . It differs from the usual entropy by a constant, i.e. $\tau(f \log f) = \log |E| - H(\frac{1}{|E|}f)$, $|E|$ is the dimension of system, $\frac{1}{|E|}f$ is a density state in the usual sense.

4.2. Capacity Bounds. The comparison property naturally generalizes to various capacities of quantum channels. We first recall the definitions.

Let $\mathcal{N} : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$ be a quantum channel and let $V \in \mathbb{B}(A', B \otimes E)$ be its Stinespring isometry. The quantum capacity $Q(\mathcal{N})$ describes the ultimate rate of qubit transmission over \mathcal{N} (without pre-shared entanglement or classical feedback). A quantum code \mathcal{C} over a channel \mathcal{N} is a triple

$$\mathcal{C} = (m, \mathcal{E}, \mathcal{D}) ,$$

which consists of an encoding $\mathcal{E} : M_m \rightarrow \mathbb{B}(A')$ and a decoding $\mathcal{D} : \mathbb{B}(B) \rightarrow M_m$ as completely positive trace preserving maps. $|\mathcal{C}| = m$ is the size of the code. The quantum communication fidelity \mathcal{F}_Q of the code \mathcal{C} is defined by

$$\mathcal{F}_Q(\mathcal{C}, \mathcal{N}) = \langle \psi_m | id_m \otimes (\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})(|\psi_m\rangle\langle\psi_m|) | \psi_m \rangle ,$$

where $\psi_m = \frac{1}{m} \sum_{i,j} e_{ij} \otimes e_{ij}$ is the maximally entangled state on $M_m \otimes M_m$. A rate triple (n, R, ϵ) consists of the number n of channel uses, the rate R of transmission and the error $\epsilon \in [0, 1]$. We say a rate triple (n, R, ϵ) is achievable on \mathcal{N} for quantum communication if there exists a quantum code \mathcal{C} of $\mathcal{N}^{\otimes n}$ for some n such that

$$\frac{\log m}{n} \geq R \quad \text{and} \quad \mathcal{F}_Q(\mathcal{C}, \mathcal{N}^{\otimes n}) \geq 1 - \epsilon.$$

Then quantum capacity $Q(\mathcal{N})$ is defined as

$$Q(\mathcal{N}) = \limsup_{\epsilon \rightarrow 0} \sup_n \{R \mid (n, R, \epsilon) \text{ achievable on } \mathcal{N} \text{ for quantum communication}\} .$$

Similarly, one can define the classical capacity $C(\mathcal{N})$ and private classical capacity $P(\mathcal{N})$. The classical capacity $C(\mathcal{N})$ considers the largest rate of classical bits that the channel \mathcal{N} can reliably transmit from Alice to Bob, while the private capacity $P(\mathcal{N})$ in addition requires complete privacy against a potential eavesdropper from the environment. A classical communication code $\mathcal{C} = (m, \mathcal{E}, \mathcal{D})$ consists of an encoding classical to quantum channel $\mathcal{E} : l_1^m \rightarrow B(H_B)$ and a decoding quantum to classical measurement $\mathcal{D} : \mathbb{B}(B) \rightarrow l_1^m$. Here the space l_1^m represents m -dimensional classical system and m is the size of the code \mathcal{C} . We denote the identity map on l_1^m as id_m . Recall that for general mixed states ρ and σ , the fidelity is

$$\mathcal{F}(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1 .$$

The classical communication fidelity \mathcal{F}_C and the private communication fidelity \mathcal{F}_P of a code \mathcal{C} are given by

$$\begin{aligned} \mathcal{F}_C(\mathcal{C}, \mathcal{N}) &= \mathcal{F}(id_m \otimes (\mathcal{D} \circ \mathcal{N} \circ \mathcal{E})(\phi_m), \phi_m) , \\ \mathcal{F}_P(\mathcal{C}, \mathcal{N}) &= \max_{\rho} \mathcal{F}(id_m \otimes \mathcal{D}(1 \otimes V(id_m \otimes \mathcal{E}(\phi_m))V^* \otimes 1), \phi_m \otimes \rho^E) , \end{aligned}$$

where $\phi_m = \frac{1}{m} \sum_i e_{ii} \otimes e_{ii}$ is the maximal correlated state and in \mathcal{F}_P the maximum runs over states ρ^E on E . The achievable triple for classical and private communication are defined similarly as for the quantum capacity but now with fidelity \mathcal{F}_C and respectively \mathcal{F}_P . Then the classical capacity $C(\mathcal{N})$ and the private classical capacity $P(\mathcal{N})$ are

$$\begin{aligned} C(\mathcal{N}) &= \limsup_{\epsilon \rightarrow 0} \sup_n \{R \mid (n, R, \epsilon) \text{ is achievable on } \mathcal{N} \text{ for classical communication}\} , \\ P(\mathcal{N}) &= \limsup_{\epsilon \rightarrow 0} \sup_n \{R \mid (n, R, \epsilon) \text{ is achievable on } \mathcal{N} \text{ for private communication}\} . \end{aligned}$$

The entanglement-assisted classical capacity C_{EA} considers the improved rate with the assistance of (unlimited) pre-generated bipartite entanglement shared by Alice and Bob. (We refer to [39] for the definition of C_{EA} and more detailed discussions about C , P and Q .)

Thanks to the capacity theorems proved by Holevo [21, 22], Schumacher and Westmoreland [34], Bennett et al [3], Lloyd [28], Shor [35] and Devetak [8], these operationally defined capacities are mathematically expressed with entropic information measures,

$$\begin{aligned} C(\mathcal{N}) &= \lim_{k \rightarrow \infty} \frac{1}{k} \chi(\mathcal{N}^{\otimes k}), \quad \chi(\mathcal{N}) = \max_{\rho^{XA'}} I(X; B)_\omega; \\ Q(\mathcal{N}) &= \lim_{k \rightarrow \infty} \frac{1}{k} Q^{(1)}(\mathcal{N}^{\otimes k}), \quad Q^{(1)}(\mathcal{N}) = \max_{\rho^{AA'}} I(A; B)_\omega; \\ P(\mathcal{N}) &= \lim_{k \rightarrow \infty} \frac{1}{k} P^{(1)}(\mathcal{N}^{\otimes k}), \quad P^{(1)}(\mathcal{N}) = \max_{\rho^{XA'}} I(X; B)_\omega - I(X; E)_\omega; \\ C_{EA}(\mathcal{N}) &= \max_{\rho^{AA'}} I(A; B)_\omega, \end{aligned}$$

where the maximums in C_{EA} and $Q^{(1)}$ run over bipartite input states $\rho^{AA'}$ and in χ and $P^{(1)}$ considers classical-quantum $\rho^{XA'}$. ω always denotes the output of ρ . In the four capacities above, only C_{EA} admits a single-letter expression. The other three involve with the limits—the *regularization* over many uses of the channel. The regularization are unavoidable due to the *superadditivity* of the “one-shot” expressions χ , $Q^{(1)}$, $P^{(1)}$. For instance, $\chi(\mathcal{N} \otimes \mathcal{M})$ may exceed $\chi(\mathcal{N}) + \chi(\mathcal{M})$ for some \mathcal{N} and \mathcal{M} [19]. (χ can be replaced by $Q^{(1)}$ and $P^{(1)}$, see [37, 36, 7, 27, 13].) $\chi(\mathcal{N} \otimes \mathcal{M}) \geq \chi(\mathcal{N}) + \chi(\mathcal{M})$ holds always and a strict inequality implies that some entangled encoding and decoding scheme is better than all nonentangled ones. Motivated by this, Winter and Yang in [45] introduced the potential capacities $\chi^{(p)}$, $Q^{(p)}$, $P^{(p)}$ as follows,

$$\begin{aligned} \chi^{(p)}(\mathcal{N}) &= \sup_{\mathcal{M}} \chi(\mathcal{N} \otimes \mathcal{M}) - \chi(\mathcal{M}), \quad Q^{(p)}(\mathcal{N}) = \sup_{\mathcal{M}} Q^{(1)}(\mathcal{N} \otimes \mathcal{M}) - Q^{(1)}(\mathcal{M}), \\ P^{(p)}(\mathcal{N}) &= \sup_{\mathcal{M}} P^{(1)}(\mathcal{N} \otimes \mathcal{M}) - P^{(1)}(\mathcal{M}), \end{aligned}$$

where the supremums runs over all channels \mathcal{M} . Note that here we use different notations from [45] to save the subscript “ p ” for L_p -norms and Rényi-type expressions. The potential capacity is always an upper bound for corresponding capacity and hence the one-shot expression.

Proposition 4.3. χ , $Q^{(1)}$ and $P^{(1)}$ and their potential analogs are convex functions over channels.

Proof. We provide a uniform argument using heralded channels. Given two channels $\mathcal{N} : \mathbb{B}(A') \rightarrow \mathbb{B}(B_1)$ and $\mathcal{M} : \mathbb{B}(A') \rightarrow \mathbb{B}(B_2)$ with common input space, let us define the

heralded channel $\Phi_\lambda : \mathbb{B}(A') \rightarrow \mathbb{B}(B_1 \oplus B_2)$ with a probability $\lambda \in [0, 1]$,

$$\Phi_\lambda(\rho) = \lambda \mathcal{N}(\rho) \oplus (1 - \lambda) \mathcal{M}(\rho) := \begin{bmatrix} \lambda \mathcal{N}(\rho) & 0 \\ 0 & (1 - \lambda) \mathcal{M}(\rho) \end{bmatrix}.$$

The output signal is heralded because Bob knows which channel is used by measuring the corresponding block. Because of the block diagonal structure, it is not hard to see that

$$\begin{aligned} Q^{(1)}(\Phi_\lambda) &= \max_{\rho^{AA'}} \lambda I_c(A)_{B_1} + (1 - \lambda) I_c(A)_{B_2}, \\ \chi(\Phi_\lambda) &= \max_{\rho^{XA'}} \lambda I(X; B_1) + (1 - \lambda) I(X; B_2). \end{aligned}$$

Note that the complementary channel of a heralded channel is again a heralded channel of complementary channels, i.e. $\Phi_\lambda^E(\rho) = \lambda \mathcal{N}^E(\rho) \oplus (1 - \lambda) \mathcal{M}^E(\rho)$. Then a similar formula holds for one-shot private capacity $P^{(1)}$,

$$P^{(1)}(\Phi_\lambda) = \max_{\rho^{XA'}} \lambda (I(X; B_1) - I(X; E_1)) + (1 - \lambda) (I(X; B_2) - I(X; E_2)).$$

Now if \mathcal{N} and \mathcal{M} have the same output space $B_1 = B_2$, then the convex combination $\lambda \mathcal{N} + (1 - \lambda) \mathcal{M}$ can be factorized through the heralded channel Φ_λ via a partial trace map. Therefore by data processing,

$$\begin{aligned} Q^{(1)}(\lambda \mathcal{N} + (1 - \lambda) \mathcal{M}) &\leq Q^{(1)}(\Phi_\lambda) = \max_{\rho^{AA'}} \lambda I_c(A)_{B_1} + (1 - \lambda) I_c(A)_{B_2} \\ &\leq \lambda Q^{(1)}(\mathcal{N}) + (1 - \lambda) Q^{(1)}(\mathcal{M}). \end{aligned}$$

Here the $Q^{(1)}$ can be replaced by χ and $P^{(1)}$. Moreover, the convexity of potential capacities follow from the convexity of their “one-shot” expressions. \blacksquare

The next theorem provides the comparison property for TRO channels. This is the analog of Corollary 3.4 in [15].

Corollary 4.4. *Let \mathcal{N} be a channel and f be an admissible symbol for \mathcal{N} . Then,*

- i) $C(\mathcal{N}) \leq C(\mathcal{N}_f) \leq C(\mathcal{N}) + \tau(f \log f)$;
- ii) $Q(\mathcal{N}) \leq Q(\mathcal{N}_f) \leq Q(\mathcal{N}) + \tau(f \log f)$;
- iii) $P(\mathcal{N}) \leq P(\mathcal{N}_f) \leq P(\mathcal{N}) + \tau(f \log f)$;
- iv) $C_{EA}(\mathcal{N}) \leq C_{EA}(\mathcal{N}_f) \leq C_{EA}(\mathcal{N}) + \tau(f \log f)$.

For i), ii) and iii), the capacity can be replaced by corresponding one-shot expression and potential capacity.

Proof. The inequalities for $\chi, Q^{(1)}$ and C_{EA} follows from Corollary 4.1 by taking maximum over all possible inputs. Note that the “one-shot” private capacity can be rewritten as

$$P^{(1)}(\mathcal{N}) = \max_{\rho^{XA'A}} I_c(A)_{B_\omega} - \sum_x p(x) I_c(A)_{B_{\omega_x}},$$

where the maximum runs over all states

$$\rho^{XA'A} = \sum_x p(x) |x\rangle\langle x| \otimes \rho_x^{AA'}$$

and $\rho_x^{AA'}$ are pure states. The coherent information is for the output $\omega_x^{AB} = id_A \otimes \mathcal{N}(\rho_x^{AA'})$ and $\omega^{AB} = id_A \otimes \mathcal{N}(\tilde{\rho}^{AA'})$ where $\tilde{\rho}^{AA'}$ is any purification of ρ^A (so $\tilde{\rho}^{AA'}$ may not be the reduced density of $\rho^{XA'A}$). Applying Corollary 4.1 ii) one have

$$I_c(A)B)_{\omega_f} - \sum_x p(x) I_c(A)B)_{\omega_{f,x}} \leq I_c(A)B)_{\omega} - \sum_x p(x) I_c(A)B)_{\omega_x} + \tau(f \log f) .$$

Then the upper bound of $P^{(1)}(\mathcal{N}_f)$ follows and the lower bound is a consequence of the lifting property $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \mathcal{N}$. For the regularization, note that by Lemma 3.7, $f^{\otimes k}$ is an admissible symbol for $\mathcal{N}^{\otimes k}$. Therefore we have

$$\begin{aligned} P(\mathcal{N}_f) &= \lim_{k \rightarrow \infty} \frac{1}{k} P^{(1)}(\mathcal{N}_f^{\otimes k}) = \lim_{k \rightarrow \infty} \frac{1}{k} P^{(1)}(\mathcal{N}_{f^{\otimes k}}^{\otimes k}) \leq \lim_{k \rightarrow \infty} \frac{1}{k} [P^{(1)}(\mathcal{N}^{\otimes k}) + \tau^k(f^{\otimes k} \log f^{\otimes k})] \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} (P^{(1)}(\mathcal{N}^{\otimes k}) + k\tau(f \log f)) = \lim_{k \rightarrow \infty} \frac{1}{k} P^{(1)}(\mathcal{N}^{\otimes k}) + \tau(f \log f) = P(\mathcal{N}) + \tau(f \log f) . \end{aligned}$$

Similarly, for the potential capacities, we use that $\mathcal{M} \otimes \mathcal{N}_f = (\mathcal{M} \otimes \mathcal{N})_{1 \otimes f}$ for an arbitrary channel \mathcal{M} and $\tau((1 \otimes f) \log 1 \otimes f) = \tau(f \log f)$. The arguments for classical capacity and quantum capacity are the same. \blacksquare

The estimates above are bounded uniformly by the term $\tau(f \log f)$. It indicates that the channels \mathcal{N}_f with admissible symbol f close to 1 are nice perturbations of \mathcal{N} .

The next proposition considers the special case when the Stinespring space is indeed a TRO. This corresponds to a particular class of channels. Recall that a channel \mathcal{N} is strongly additive for χ (respectively, $Q^{(1)}$ and $P^{(1)}$) if $\chi(\mathcal{N}) = \chi^{(p)}(\mathcal{N})$ (respectively, $Q^{(1)}(\mathcal{N}) = Q^{(p)}(\mathcal{N})$ and $P^{(1)}(\mathcal{N}) = P^{(p)}(\mathcal{N})$). This means $\chi(\mathcal{N} \otimes \mathcal{M}) = \chi(\mathcal{N}) + \chi(\mathcal{M})$ for any \mathcal{M} and hence $\chi(\mathcal{N}) = C(\mathcal{N})$.

Proposition 4.5. *Let $\mathcal{N}(|x\rangle\langle y|) = xy^*$, $x, y \in X$ be the TRO channel associated with X . Then*

- i) \mathcal{N} is a direct sum of partial traces;
- ii) \mathcal{N} is strongly additive for $\chi, Q^{(1)}$ and $P^{(1)}$.

Assume that $X \cong \oplus_i M_{n_i, M_i}$, then

$$Q^{(1)}(\mathcal{N}) = P^{(1)}(\mathcal{N}) = \log \max_i n_i, \chi(\mathcal{N}) = \log(\sum_i n_i), C_{EA}(\mathcal{N}) = \log(\sum_i n_i^2) .$$

Proof. According to the direct sum structure of X , we can decompose X as following,

$$X = \oplus_i X_i, X_i \cong M_{n_i, m_i},$$

and X_i are mutually orthogonal subspaces in X . Thus the channel \mathcal{N} can be rewritten as

$$\mathcal{N}(|x\rangle\langle y|) = xy^* = \oplus_i x_i y_i^*, \quad |x\rangle = \oplus_i |x_i\rangle, \quad |y\rangle = \oplus_i |y_i\rangle.$$

It is sufficient to see on each subspace X_i \mathcal{N} is a partial trace. Indeed, by identifying $X_i \cong M_{n_i, m_i}$ as Hilbert spaces, we know from (2.2)

$$\mathcal{N}_i(|x_i\rangle\langle y_i|) = x_i y_i^* = id_{n_i} \otimes tr_{m_i}(|x_i\rangle\langle y_i|).$$

The capacity formulae are easy applications of Proposition 1 in [14]. ■

Remark 4.6. To be precise, X may be of the form $\oplus_i M_{n_i, m_i} \otimes \mathbb{C}1_{l_i}$ with the multiplicity l_i for i -th block. In this situation, each direct summand \mathcal{N}_i is a “generalized” partial trace as follows

$$\mathcal{N}_i(\rho_i) = (id_{n_i} \otimes tr_{m_i}(\rho_i)) \otimes \pi_{l_i},$$

where $\pi_{l_k} = \frac{1}{l_k} 1_{l_k}$ is the l_k -dimensional completely mixed state. Namely, each \mathcal{N}_i is a partial trace plus a dummy state π_{l_k} . The channel $\mathcal{N} = \oplus_i \mathcal{N}_i$ here is equivalent to the one in Proposition 4.5 without redundancy, in the sense that they can factor through each other. Hence for capacity purpose we can identify these channels. We will continue using the simpler identification $X = \oplus_i M_{n_i, m_i}$ in the following.

The next theorem generalizes the negative cb -entropy formula in [15]. The negative cb -entropy $-S_{cb}$ of a channel \mathcal{N} is defined as

$$-S_{cb}(\mathcal{N}) = \sup_{\rho \text{ pure}} H(A)_\omega - H(AB)_\omega,$$

where $\omega^{AB} = id_A \otimes \mathcal{N}(\rho^{AA'})$. It was first introduced in [9], characterized as the derivative of the completely bounded $1 \rightarrow p$ norm at $p = 1$,

$$-S_{cb}(\mathcal{M}) = \frac{d}{dp} \Big|_{p=1} \|\mathcal{M} : S_1(A') \rightarrow S_p(B)\|_{cb}, \quad (4.1)$$

and later rediscovered as “reverse coherent information” with an operational meaning in [16]. In the following, we use the short notation $|A| = \dim A$ for the dimension of a Hilbert space.

Theorem 4.7. *Let \mathcal{N} be a quantum channel and f be an admissible symbol for \mathcal{N} . Suppose that the complementary channel $\mathcal{N}^E : B(A') \rightarrow B(E)$ is unital up to a scalar, $\mathcal{N}^E(1_{A'}) = \frac{|A'|}{|E|} 1_E$. Then*

$$S_{cb}(\mathcal{N}_f) = \log \frac{|A'|}{|E|} + \tau(f \log f).$$

Proof. Let $A \cong A'$ and denote e_{ij} the matrix units in $\mathbb{B}(A) \cong M_m$ for $m = |A'|$. Let $\{|h_i\rangle\}$ be an orthonormal basis of A' . For a channel $\mathcal{M} : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$, its Choi matrix $J_{\mathcal{M}}$ is given by

$$J_{\mathcal{M}} = \sum_{i,j} e_{ij} \otimes \mathcal{M}(|h_i\rangle\langle h_j|) \in \mathbb{B}(A \otimes B).$$

The completely bounded $1 \rightarrow p$ norm of a map \mathcal{M} is same with the vector-valued (∞, p) norm (defined in (2.4)) of its Choi matrix $J_{\mathcal{M}}$ (see e.g. [33, 12]),

$$\|\mathcal{M} : S_1(A') \rightarrow S_p(B)\|_{cb} = \|J_{\mathcal{M}}\|_{S_{\infty}(A, S_p(B))}.$$

In particular, for $p = \infty$, $S_{\infty}(A, S_{\infty}(B)) = \mathbb{B}(A \otimes B)$. The Choi matrix of \mathcal{N}_f is given by

$$\sum_{i,j} e_{i,j} \otimes \mathcal{N}_f(|h_i\rangle\langle h_j|) = \sum_{i,j} e_{i,j} \otimes (\widehat{V h_i f V h_j}^*) = B(e_{11} \otimes f)B^* \in \mathbb{B}(A \otimes B),$$

where $B = \sum_i e_{i1} \otimes \widehat{V h_i}$. Since \mathcal{N}^E is unital up to a scale,

$$\mathcal{N}^E(\sum_i |h_i\rangle\langle h_i|) = \sum_i \widehat{V h_i}^* \widehat{V h_i} = \frac{|A'|}{|E|} 1_E.$$

This implies that $(\frac{|E|}{|A'|})^{\frac{1}{2}} B$ is an isometry. We can define the following $*$ -homomorphism

$$\pi(f) = \frac{|E|}{|A'|} B(e_{11} \otimes f)B^*.$$

Note that $tr(f) = tr(\pi(f))$, then

$$\|f\|_{p,\tau}^p = |E|^{-1} tr_E(|f|^p) = |E|^{-1} tr_{AB}(\pi(|f|^p)) = |E|^{-1} \|\pi(|f|)\|_p^p.$$

Therefore we get

$$\|f\|_{p,\tau} = |E|^{-1/p} \|\pi(f)\|_p = |E|^{-1/p} \frac{|E|}{|A'|} \|B(e_{11} \otimes f)B^*\|_p = |E|^{1-1/p} |A'|^{-1} \|J_{\mathcal{N}_f}\|_p.$$

By the definition (2.3), we obtain a lower bound for the $S_{\infty}(A', S_p(B))$ norm,

$$\|J_{\mathcal{N}_f}\|_{S_{\infty}(A', S_p(B))} \geq |A'|^{-1/p} \|J_{\mathcal{N}_f}\|_p = \frac{|A'|^{1-1/p}}{|E|} \|f\|_{p,\tau}. \quad (4.2)$$

For the upper bound, note that π is a $*$ -homomorphism, then

$$\|\pi(f)\|_{\infty} = \left\| \frac{|E|}{|A'|} J_{\mathcal{N}_f} \right\|_{\infty} = \frac{|E|}{|A'|} \|\mathcal{N}_f : S_1(A') \rightarrow S_{\infty}(B)\|_{cb}. \quad (4.3)$$

Now assume that X is a TRO containing $X^{\mathcal{N}}$ and $N \in \mathbb{B}(E)$ be a $*$ -subalgebra containing f and independent of $E(X)$. For any admissible symbol $g \in N$, \mathcal{N}_g satisfies that

$$tr(\mathcal{N}_g(\rho)) = \tau(g) tr(\mathcal{N}(\rho)).$$

Thus $\|\mathcal{N}_g : S_1(A') \rightarrow S_1(B)\|_{cb} = |\tau(g)| \leq \|g\|_{1,\tau}$. we have

$$\|\pi : L_1(N, \tau) \rightarrow S_\infty(A, S_1(B))\| \leq \frac{|E|}{|A'|}$$

By Stein's interpolation theorem (3.1), we deduce that

$$\|\pi : L_p(N, \tau) \rightarrow S_\infty(A, S_p(B))\| \leq \left(\frac{|E|}{|A'|}\right)^{1/p}. \quad (4.4)$$

Combining (4.4) with (4.2), the upper and lower bound coincide and give

$$\|J_{\mathcal{N}_f}\|_{S_\infty(A, S_p(B))} = \left(\frac{|A'|}{|E|}\right)^{1-1/p} \|f\|_{p,\tau}.$$

Note that for all $1 \leq p \leq \infty$, the maximal entangled state is a norm attaining element. The assertion follows by differentiating the above equality at $p = 1$. \blacksquare

4.3. The capacity regions. The capacity regions of a quantum channel consider the trade off of different resources in quantum information theory. Based on research due to Devetak and Shor [10], Abeyesinghe et al [1], Collins and Popescu [6] and many others, Hsieh and Wilde introduced the two kinds of capacity regions: the quantum dynamic region C_{CQE} and private dynamic region C_{RPS} . The quantum dynamic region C_{CQE} considers a combined version of classical communication “C”, quantum communication “Q” and entanglement generation “E”, while the private dynamic region C_{RPS} , with the idea of the Collins-Popescu analogy [6], unifies the public classical communication “R”, private classical communication “P” and secret key distribution “S”. We refer to their papers [41, 40] for the operational definitions of C_{CQE} and C_{RPS} . Here we state the capacity region theorems from [41, 40] for the convenience of readers.

Let $\mathcal{N} : \mathbb{B}(A') \rightarrow \mathbb{B}(B)$ be a quantum channel and $V : A' \rightarrow B \otimes E$ be its Stinespring isometry $V : A' \rightarrow B \otimes E$. The quantum dynamic region $C_{CQE}(\mathcal{N})$ is characterized as follows,

$$C_{CQE}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} C_{CQE}^{(1)}(\mathcal{N}^{\otimes k})}, \quad C_{CQE}^{(1)} \equiv \bigcup_{\omega} C_{CQE,\omega}^{(1)},$$

where the overbar represents the closure of a set. The “one-shot” region $C_{CQE}^{(1)} \subset \mathbb{R}^3$ is the union of the “one-shot, one-state” regions $C_{CQE,\omega}^{(1)}$, which are the sets of all rate triples (C, Q, E) such that:

$$C + 2Q \leq I(AX; B)_\omega, \quad Q + E \leq I(A)BX)_\omega, \quad C + Q + E \leq I(X; B)_\omega + I(A)BX)_\omega.$$

The above entropy quantities are with respect to a classical-quantum state

$$\omega^{XABE} = \sum_x p_X(x) |x\rangle\langle x|^X \otimes (1_A \otimes V) \rho_x^{AA'} (1_A \otimes V^*)$$

and the states $\rho_x^{AA'}$ are pure. Similarly, the private dynamic region is given by,

$$C_{RPS}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} C_{RPS}^{(1)}(\mathcal{N}^{\otimes k})}, \quad C_{RPS}^{(1)} \equiv \bigcup_{\omega} C_{RPS,\omega}^{(1)}.$$

The "one-shot, one-state" region $C_{RPS}^{(1)}(\mathcal{N}) \subset \mathbb{R}^3$ is the set of all triples (R, P, S) such that

$$\begin{aligned} R + P &\leq I(YX; B)_{\omega}, \quad P + S \leq I(Y; B|X)_{\omega} - I(Y; E|X)_{\omega}, \\ R + P + S &\leq I(YX; B)_{\omega} - I(Y; E|X)_{\omega}. \end{aligned}$$

The above entropic quantities are with respect to a classical-quantum state ω^{XYBE} where

$$\omega^{XYBE} \equiv \sum_x p_{X,Y}(x, y) |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y \otimes (V \rho_{x,y}^{A'} V^*).$$

In general it is difficult to completely describe the capacity regions. Nevertheless, there is a mathematically nice way to characterize the "one-shot, one-state" region $C_{CQE,\omega}^{(1)}$ and $C_{RPS,\omega}^{(1)}$. Let us consider two cones,

$$W_1 = \{(C, Q, E) | 2Q + C \leq 0, Q + E \leq 0, Q + E + C \leq 0\}$$

and

$$W_2 = \{(R, P, S) | R + P \leq 0, P + S \leq 0, R + P + S \leq 0\}.$$

The first one is the resource trading off via teleportation, superdense coding and entanglement distribution and the second is the cone obtained from secret key distribution, the one-time pad and private-to-public transmission. We will follow the approach in our previous work to compare the rate triple $(I(X; B)_{\omega}, \frac{1}{2}I(A; B|X)_{\omega}, -\frac{1}{2}I(A; E|X)_{\omega})$ for each single input state $\rho^{XA'}$ and respectively $(I(X; B)_{\omega}, I(Y; B|X)_{\omega}, -I(Y; E|X)_{\omega})$ for each $\rho^{XYA'}$.

Example 4.8. Let \mathcal{N} be the TRO channel associated with $X \cong M_{n_i, m_i}$. We know from Proposition 4.5 that

$$\mathcal{N} = \oplus_i id_{n_i} \otimes tr_{m_i}$$

as a direct sum of partial traces. The capacity regions of this class of channels are accessible. The quantum dynamic region regularizes $C_{CQE}(\mathcal{N}) = C_{CQE}^{(1)}(\mathcal{N})$, and it is characterized as a union of the followings

$$\begin{aligned} C + 2Q &\leq H(\{p_{\lambda,\mu}(i)\}) + 2 \sum_i p_{\lambda,\mu}(i) \log n_i, \quad Q + E \leq \sum_i p_{\lambda,\mu}(i) \log n_i, \\ C + Q + E &\leq H(\{p_{\lambda,\mu}(i)\}) + \sum_i p_{\lambda,\mu}(i) \log n_i. \end{aligned}$$

for all $\lambda, \mu \geq 0$. Here $\{p_{\lambda,\mu}(i)\}$ is the probability distribution given by

$$p_{\lambda,\mu}(i) = n_i^{\frac{2+\lambda+\mu}{\mu+1}} / \left(\sum_i n_i^{\frac{2+\lambda+\mu}{\mu+1}} \right).$$

Similarly, for the public-private dynamic region, $C_{RPS}(\mathcal{N}) = C_{RPS}^{(1)}(\mathcal{N})$ is the union of

$$\begin{aligned} R + P &\leq H(\{p'_{\lambda,\mu}(i)\}) + \sum_i p'_{\lambda,\mu}(i) \log n_i, \quad P + S \leq \sum_i p'_{\lambda,\mu}(i) \log n_i, \\ R + P + S &\leq H(\{p'_{\lambda,\mu}(i)\}) + \sum_i p'_{\lambda,\mu}(i). \end{aligned}$$

for all $\lambda, \mu \geq 0$. $\{p'_{\lambda,\mu}(i)\}$ is the probability distribution given by

$$p'_{\lambda,\mu}(i) = n_i^{\frac{1+\lambda+\mu}{\mu+1}} / \left(\sum_i n_i^{\frac{1+\lambda+\mu}{\mu+1}} \right).$$

Corollary 4.9. *Let \mathcal{N} be a channel and f be an admissible symbol for \mathcal{N} . Denote $\tau = \tau(f \log f)$. Then*

- i) $C_{CQE}(\mathcal{N}) \subset C_{CQE}(\mathcal{N}_f) \subset C_{CQE}(\mathcal{N}) + (\tau, \frac{\tau}{2}, \frac{\tau}{2})$;
- ii) $C_{RPS}(\mathcal{N}) \subset C_{RPS}(\mathcal{N}_f) \subset C_{RPS}(\mathcal{N}) + (\tau, \tau, \tau)$.

Proof. The argument for the two kinds of regions are similar. One can see Proposition 3.7 in [15] for a identical proof for quantum dynamic region C_{CQE} . Here we give the proof for the private dynamic region C_{RPS} . Let us assume

$$\omega_f^{XYABE} = \sum_x p_{X,Y}(x, y) |x\rangle\langle x|^X \otimes |y\rangle\langle y|^Y \otimes (1 \otimes V_f)(\rho_{x,y}^{A'A})(1 \otimes V_f^*),$$

where $\rho_{x,y}^{A'A}$ are pure states. We denote (R^f, P^f, S^f) for the rate triple

$$(I(X; B)_{\omega_f}, I(Y; B|X)_{\omega_f}, -I(Y; E|X)_{\omega_f}).$$

By the entropic inequality (4.1),

$$I(X; B)_{\omega_f} \leq \tau(f \log f) + I(X; B)_{\omega_1}.$$

From this, we may assume $R^f = R^1 + \tau(f \log f) - \alpha_1$ for some $\alpha_1 \geq 0$. Similarly, we have

$$\begin{aligned} I(Y; B|X)_{\omega_f} &= H(Y|X)_{\omega_f} + H(B|X)_{\omega_f} - H(YB|X)_{\omega_f} \\ &= H(Y|X)_{\omega_f} + \sum_x p(x) H(\omega_{f,x}^B) - \sum_x p(x) (H(Y|X=x) + \sum_y p(y|x) H(\omega_{x,y,f}^B)) \\ &\leq I(Y; B|X)_{\omega} + \tau(f \log f), \end{aligned}$$

and

$$I(Y; E|X)_{\omega_f} = H(Y|X)_{\omega_f} + H(E|X)_{\sigma,f} - H(YE|X)_{\omega_f}$$

$$\begin{aligned}
&= H(Y|X)_{\omega_f} + \sum_x p(x) H(\omega_{f,x}^E) - \sum_x p(x) (H(Y|X=x) + \sum_y p(y|x) H(\omega_{x,y,f}^E)) \\
&\geq I(Y; E|X)_{\omega} - \tau(f \log f) .
\end{aligned}$$

This means

$$P^f = P^1 + \tau(f \log f) - \alpha_2, \quad S^f = S^1 + \tau(f \log f) - \alpha_3$$

for some $\alpha_2, \alpha_3 \geq 0$. Now it is obvious that $(-\alpha_1, -\alpha_2, -\alpha_3) \in W$, then we have

$$(R^f, P^f, S^f) \in (\tau, \tau, \tau) + (R^1, P^1, S^1) + W .$$

Taking the union for all ω , we have

$$C_{RPS}^{(1)}(\mathcal{N}_f) \in (\tau, \tau, \tau) + C_{RPS}^{(1)}(\mathcal{N}) + W + W .$$

For the cone W , we have $W + W = W$ and this concludes that

$$C_{RPS}^{(1)}(\mathcal{N}_f) \subset (\tau, \tau, \tau) + C_{RPS}^{(1)}(\mathcal{N}) .$$

For regularization, we apply the above estimates to the tensor channel

$$\begin{aligned}
\frac{1}{k} C_{RPS}^{(1)}(\mathcal{N}_f^{\otimes k}) &= \frac{1}{k} C_{RPS}^{(1)}((\mathcal{N}^{\otimes k})_{f^{\otimes k}}) \subset \frac{1}{k} (k(\tau, \tau, \tau) + C_{RPS}^{(1)}(\mathcal{N}^{\otimes k})) \\
&= (\tau, \tau, \tau) + \frac{1}{k} C_{RPS}^{(1)}(\mathcal{N}^{\otimes k}) ,
\end{aligned}$$

which completes the proof. ■

4.4. Strong converse rates. A “strong converse” means there is a sharp drop off for code fidelity above the optimal transmission rate. More generally, we will investigate rates above which the transmission only succeeds with arbitrarily small probability. We consider the strong converse of a quantum channel for classical, quantum and private communication. We say r is a strong converse rate for classical (respectively, quantum, private) communication if for every sequence of achievable triple (n, R_n, ϵ_n) of classical (respectively, quantum, private) communication, we have

$$\liminf_{n \rightarrow \infty} R_n > r \Rightarrow \lim_{n \rightarrow \infty} \epsilon_n = 1 .$$

Then the strong converse classical capacity C^\dagger , the strong converse quantum capacity Q^\dagger and the strong converse private capacity P^\dagger are defined as the infimum of corresponding strong converse rates. We say a channel \mathcal{N} has (classical, quantum or private) strong converse if the capacity equals to the strong converse capacity (respectively, $C^\dagger(\mathcal{N}) = C(\mathcal{N})$, $Q^\dagger(\mathcal{N}) = Q(\mathcal{N})$ or $P^\dagger(\mathcal{N}) = P(\mathcal{N})$).

There are known upper bounds for strong converse capacities. It is shown by Wilde et al [44] that for any channel \mathcal{N} ,

$$C^\dagger(\mathcal{N}) \leq \lim_{k \rightarrow \infty} \frac{\chi_p(\mathcal{N}^{\otimes k})}{k}, \quad \chi_\alpha(\mathcal{N}) = \max_{\rho^{XA'}} I_p(X; B)_\omega, \quad (4.5)$$

where the sandwich Rényi mutual information are given by

$$I_p(A; B)_\rho = \inf_{\sigma^B} D_p(\rho^{AB} \| \rho^A \otimes \sigma^B). \quad (4.6)$$

For the quantum strong converse, the Rains information of a quantum channel is shown to be a strong converse rate [38]. The relative entropy of entanglement $E_R(\mathcal{N})$ is a upper bound for the private strong converse capacity [42],

$$P^\dagger(\mathcal{N}) \leq E_R(\mathcal{N}), \quad E_R(\mathcal{N}) = \max_{\rho^{AA'}} E_R(\text{id}_A \otimes \mathcal{N}_f(\rho)). \quad (4.7)$$

The relative entropy of entanglement $E_R(\rho)$ for a bipartite ρ^{AB} is

$$E_R(\rho^{AB}) = \inf_{\sigma^{AB} \in S(A:B)} D(\rho^{AB} \| \sigma^{AB}),$$

where $S(A:B)$ stands for the separable states between A and B . These results in particular imply the strong converses of Hadamard channels and entanglement-breaking channels for classical, quantum and private communication. In these arguments, the sandwich Rényi relative entropy play an important role.

For $1 < p \leq \infty$ and $1/p + 1/p' = 1$, we introduce the Rényi coherent information of a channel for as an analog of (4.5)

$$Q_p^{(1)}(\mathcal{N}) = \max_{\rho^{AA'}} I_{c,p}(A; B)_\omega, \quad I_{c,p}(A; B)_\omega = p' \log \|\omega^{BA}\|_{S_1(B, S_p(A))}.$$

The following result is probably known to experts but not stated explicitly in the literature.

Proposition 4.10. *For any channel \mathcal{N} ,*

$$\limsup_{k \rightarrow \infty} \frac{Q_p^{(1)}(\mathcal{N}^{\otimes k})}{k},$$

is a quantum strong converse rate of \mathcal{N} for all $1 < p \leq \infty$.

Proof. Denote $R_p = \limsup_{k \rightarrow \infty} \frac{1}{k} Q_p^{(1)}(\mathcal{N}^{\otimes k})$. Let $m = 2^M$ and $\frac{1}{p} + \frac{1}{p'} = 1$. It is sufficient to show that for an arbitrary code $\mathcal{C} = (m, \mathcal{E}, \mathcal{D})$ of \mathcal{N} ,

$$\mathcal{F}(\mathcal{C}, \mathcal{N}) \leq m^{-\frac{1}{p'}} \exp\left(\frac{1}{p'} Q_p^{(1)}(\mathcal{N})\right).$$

Indeed, let \mathcal{C}_n be a sequence of codes such that $\liminf_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{C}_n| > R_p + \epsilon$,

$$\mathcal{F}(\mathcal{C}_n, \mathcal{N}^{\otimes n}) \leq m^{-\frac{1}{p'}} \exp\left(\frac{1}{p'} Q_p^{(1)}(\mathcal{N}^{\otimes n})\right) = \exp\left(\frac{1}{p'} (Q_p^{(1)}(\mathcal{N}^{\otimes n}) - M)\right) \leq 2^{(-\frac{1}{p'} n \epsilon)},$$

for n large enough. To prove (4.4), we define $\omega = id_m \otimes (\mathcal{D} \circ \mathcal{N} \circ \mathcal{C})(|\psi_m\rangle\langle\psi_m|)$. Then the fidelity is given by

$$\mathcal{F}(\mathcal{C}, \mathcal{N}) = tr(\omega |\psi_m\rangle\langle\psi_m|) \leq \|\omega\|_{S_1^m(S_p^m)} \|\psi_m\rangle\langle\psi_m|\|_{M_m(S_p^m)} .$$

Note that \mathcal{E} and \mathcal{D} are completely positive trace preserving maps and hence

$$\|\mathcal{E} : S_1^m \rightarrow S_1(A')\|_{cb} = 1 , \quad \|\mathcal{D} : S_1(B) \rightarrow S_1^m\|_{cb} = 1 .$$

This implies

$$\|\omega\|_{S_1^m(S_p^m)} \leq 2^{(p'Q_p^{(1)}(\mathcal{N}))} \|\psi_m\rangle\langle\psi_m|\|_{S_1^m \otimes S_1^m} = 2^{(p'Q_p^{(1)}(\mathcal{N}))} .$$

For the second term, we use the interpolation relation

$$M_m(S_{p'}^m) = [M_m(S_1^m), M_m(M_m)]_{\frac{1}{p'}} .$$

Therefore we have

$$\|\psi\rangle\langle\psi|\|_{M_m(S_{p'}^m)} = \|\psi_m\rangle\langle\psi_m|\|_{M_m(M_m)}^{\frac{1}{p}} \|\psi_m\rangle\langle\psi_m|\|_{M_m(S_1^m)}^{\frac{1}{p'}} \leq m^{-\frac{1}{p'}} , \quad (4.8)$$

where we used the fact $\|\psi_m\rangle\langle\psi_m|\|_{M_m(S_1^m)} = 1/m$. (4.8) is indeed an equality. Combining these two estimates, we obtain (4.4). \blacksquare

Example 4.11. We consider again the case when the Stinespring space $X^{\mathcal{N}}$ is a TRO space. Assume that $\mathcal{N} = \oplus_i id_{n_i} \otimes tr_{m_i}$ is a direct sum of partial traces, then it is not hard to calculate

$$\chi_p(\mathcal{N}) = \log\left(\sum_i n_i\right) , \quad Q_p^{(1)}(\mathcal{N}) = E_{R,p}(\mathcal{N}) = \max_i \log n_i .$$

Note that all these terms are additive. Let $\mathcal{M} = \oplus_j id_{n'_j} \otimes tr_{m'_j}$ be another direct sum of partial traces. Then

$$\mathcal{N} \otimes \mathcal{M} = \oplus_{i,j} id_{n_i n'_j} \otimes tr_{m_i m'_j} .$$

Apply the above formulae for $\mathcal{N} \otimes \mathcal{M}$, we obtain

$$\chi_p(\mathcal{N} \otimes \mathcal{M}) = \log\left(\sum_{i,j} n_i n'_j\right) = \log\left(\sum_i n_i\right) + \log\left(\sum_j n'_j\right) = \chi_p(\mathcal{N}) + \chi_p(\mathcal{M}) ,$$

$$Q_p^{(1)}(\mathcal{N} \otimes \mathcal{M}) = \max_{i,j} \log n_i n'_j = \max_i \log n_i + \max_j \log n'_j = Q_p^{(1)}(\mathcal{N}) + Q_p^{(1)}(\mathcal{M}) ,$$

and similarly for $E_{R,p}$. Hence the regularization is trivial. By Proposition 4.5, \mathcal{N} has strong converse for classical, quantum and private communication.

The following lemma is an analog of (4.1) for Rényi information measures.

Lemma 4.12. *Let \mathcal{N} be a channel and f be an admissible symbol for \mathcal{N} . Denote $\omega_f^{AB} = id_A \otimes \mathcal{N}_f(\rho^{AA'})$ and $\omega^{AB} = id_A \otimes \mathcal{N}(\rho^{AA'})$ respectively for a bipartite state $\rho^{AA'}$. Then the following inequalities hold:*

- i) $I_{c,p}(A)B)_{\omega_1} \leq I_{c,p}(A)B)_{\omega_f} \leq I_{c,p}(A)B)_{\omega_1} + p' \log \|f\|_{p,\tau}$;
- ii) $I_p(A : B)_{\omega_1} \leq I_p(A : B)_{\omega_f} \leq I_p(A : B)_{\omega_1} + p' \log \|f\|_{p,\tau}$;
- iii) $E_{R,p}(\omega_1) \leq E_{R,p}(\omega_f) \leq E_{R,p}(\omega_1) + p' \log \|f\|_{p,\tau}$.

Proof. Let X be a TRO such that \mathcal{N} 's Stinespring space $X^{\mathcal{N}} \subset X$ and $C^*(f)$ is independent of $E(X)$. All lower bounds follows from the factorization property $\mathcal{E}_{B(X)} \circ \mathcal{N}_f = \mathcal{N}_1$, where $\mathcal{E}_{B(X)}$ is the conditional expectation from $\mathbb{B}(B)$ onto the right algebra $B(X)$. The upper estimate of i) is a direct consequence of the vector-valued $(1, p)$ norm inequality in Corollary 3.8. Indeed,

$$\begin{aligned} I_{c,p}(A)B)_{\omega_f} &= p' \log \|\omega_f\|_{S_1(B, S_p(A))} \leq p' \log \|f\|_{\tau,p} \|\omega\|_{S_1(B, S_p(A))} \\ &\leq p' \log \|f\|_{\tau,p} + p' \log \|\omega\|_{S_1(B, S_p(A))} \leq p' \log \|f\|_{\tau,p} + I_{c,p}(A)B)_{\omega} . \end{aligned}$$

For ii), note that $\mathcal{E}_{B(X)} \circ \mathcal{N} = \mathcal{N}$,

$$id_A \otimes \mathcal{E}_{B(X)}(\omega) = id_A \otimes (\mathcal{E}_{B(X)} \circ \mathcal{N})(\rho) = id_A \otimes \mathcal{N}(\rho) = \omega .$$

Therefore for the Rényi mutual information,

$$\begin{aligned} I_p(A; B)_{\omega} &= \inf_{\sigma^B} D_p(\omega \| \omega^A \otimes \sigma^B) \geq \inf_{\sigma^B} D_p(\omega \| \omega^A \otimes \mathcal{E}_{B(X)}(\sigma^B)) \\ &\geq \inf_{\sigma^B \in B(X)} D_p(\omega \| \omega^A \otimes \sigma^B) \geq \inf_{\sigma^B} D_p(\omega \| \omega^A \otimes \sigma^B) . \end{aligned}$$

Hence $I_p(A; B)_{\omega} = \inf_{\sigma^B \in B(X)} D_p(\omega \| \omega^A \otimes \sigma^B)$. Combined with the “local comparison property” (3.4), we have

$$\begin{aligned} I_p(A; B)_{\omega_f} &= \inf_{\sigma^B} D_p(\omega_f \| \omega^A \otimes \sigma^B) \leq \inf_{\sigma^B \in B(X)} D_p(\omega_f \| \omega^A \otimes \sigma^B) \\ &\leq \inf_{\sigma^B \in B(X)} D_p(\omega \| \omega^A \otimes \sigma^B) + p' \log \|f\|_{\tau,p} = I_p(A; B)_{\omega} + p' \log \|f\|_{\tau,p} . \end{aligned}$$

The upper bounds for Rényi relative entropy of entanglement $E_{R,p}$ is similar. Note that for a separable state $\sigma^{AB} = \sum_i p(i) \sigma_i^A \otimes \sigma_i^B$,

$$id_A \otimes \mathcal{E}_{B(X)}(\sigma^{AB}) = \sum_i p(i) \sigma_i^A \otimes \mathcal{E}_{B(X)}(\sigma_i^B)$$

is again a separable state in $\mathbb{B}(A) \otimes B(X) \subset \mathbb{B}(A \otimes B)$. Let us denote $S(A : B(X))$ for separable states in $\mathbb{B}(A) \otimes B(X)$. Then

$$\begin{aligned} E_{R,p}(\omega) &= \inf_{\sigma \in S(A:B)} D_p(\omega \| \sigma) \geq \inf_{\sigma \in S(A:B)} D_p(\omega \| id_A \otimes \mathcal{E}_{B(X)}(\sigma)) \\ &\geq \inf_{\sigma \in S(A:B(X))} D_p(\omega \| \sigma) \geq \inf_{\sigma \in S(A:B)} D_p(\omega \| \sigma) . \end{aligned}$$

Thus, $E_{R,p}(\omega) = \inf_{\sigma \in S(A:B(X))} D_p(\omega \| \sigma)$. Again by Theorem 3.4,

$$E_{R,p}(\omega_f) = \inf_{\sigma \in S(A:B)} D_p(\omega_f \| \sigma) \leq \inf_{\sigma \in S(A:B(X))} D_p(\omega_f \| \sigma)$$

$$\leq \inf_{\sigma \in S(A:B(X))} D_p(\omega || \sigma) + p' \log \|f\|_{\tau,p} = E_{R,p}(\omega) + p' \log \|f\|_{\tau,p},$$

which completes the proof. ■

The next theorem is the comparison property for strong converse rates.

Corollary 4.13. *Let \mathcal{N}_f be a X -TRO channels with symbol f . Assume that $X \cong \oplus_i M_{n_i, m_i}$, then*

- i) $\log(\sum_i n_i) \leq C^\dagger(\mathcal{N}_f) \leq \log(\sum_i n_i) + \tau(f \log f)$;
- ii) $\max_i \log n_i \leq Q^\dagger(\mathcal{N}_f) \leq \max_i \log n_i + \tau(f \log f)$;
- ii) $\max_i \log n_i \leq P^\dagger(\mathcal{N}_f) \leq \max_i \log n_i + \tau(f \log f)$.

Proof. When $f = 1$ and $\mathcal{N}_f = \mathcal{N}$, the estimates correspond to the formulae given in the Example 4.11. Taking the supremum of all inputs $\rho^{XA'}$ for (4.12), we have

$$\chi_p(\mathcal{N}_f) \leq \chi_p(\mathcal{N}) + p' \log \|f\|_{\tau,p}.$$

The upper bound of $C^\dagger(\mathcal{N}_f)$ follows from regularization with the help of (4.5),

$$\begin{aligned} C^\dagger(\mathcal{N}_f) &\leq \lim_{k \rightarrow \infty} \frac{1}{k} \chi_p(\mathcal{N}_f^{\otimes k}) \leq \lim_{k \rightarrow \infty} \frac{1}{k} (\chi_p(\mathcal{N}^{\otimes k}) + p' \log \|f^{\otimes k}\|_{\tau^k,p}) \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{k} (k \chi_p(\mathcal{N}) + k p' \log \|f\|_{\tau,p}) = \log(\sum_i n_i) + p' \log \|f\|_{\tau,p} \end{aligned}$$

where we used the facts that

$$\|f^{\otimes k}\|_{\tau^k,p} = \|f\|_{\tau,p}^k \quad \text{and} \quad \chi_p(\mathcal{N}^{\otimes k}) = k \chi_p(\mathcal{N}).$$

Then taking the limit $p \rightarrow 1$ yields

$$C^\dagger(\mathcal{N}_f) \leq \log(\sum_i n_i) + \lim_{p \rightarrow 1} p' \log \|f\|_{\tau,p} = \log(\sum_i n_i) + \tau(f \log f)$$

The argument for P^\dagger and Q^\dagger follow similarly with the upper bounds (4.7) and Proposition 4.10. ■

5. EXAMPLES

5.1. Previous work. In [15] we started from quantum group channels and identified a class of channels which is accessible to our interpolation techniques. To obtain the comparison property similar to Theorem 3.4, considerable algebraic assumptions are made, based on operator algebra structure. In proof, we used intensively the Haagerup tensor product and operator space interpolation. In this new setup we find simplified assumptions and less restriction on constructing channels. Let us explain the connection.

Let (N, τ) be a finite dimensional von Neumann algebra with a normalized trace τ and let $U \in M_m \otimes N$ be an unitary. The channel $\theta_f : M_m \rightarrow M_m$ defined as

$$\theta_f(\rho) = id \otimes \tau(U(\rho \otimes f)U^*)$$

is a VN-channel (von Neumann algebra channel) if N and U satisfy the following conditions:

- C1) there exists a standard inclusion $M \subset M_m$ of a $*$ -subalgebra M ;
- C2) U admits a tensor representation $U = \sum_i x_i \otimes y_i$ with $x_i \in M'$, $y_i \in N$;
- C3) the operator $\mathcal{B} = \sum_i |x_i\rangle \otimes \langle y_i^*| \in \mathbb{B}(L_2(N, \tau), L_2(M, tr))$ satisfies $\mathcal{B}\mathcal{B}^* = id_{L_2(M)}$.

Here M' is the commutant of M and $L_2(N, \tau), L_2(M, tr)$ are the GNS-construction spaces (see [15] for detailed definitions). The standard inclusion in the first condition C1 means M is isomorphic to $\oplus_k M_{n_k} \otimes 1_{n_k}$ as a subalgebra M_m , and as a consequence we have $L_2(M, tr) \cong H_m$, the m -dimensional Hilbert space. The Stinespring isometry of θ_f can be written as,

$$W_f(|h\rangle) = \left(\sum_i x_i \otimes \lambda(y_i) \right) |h\rangle \otimes |f^{\frac{1}{2}}\rangle.$$

where $\lambda(y_i)$ is the GNS representation and $|f^{\frac{1}{2}}\rangle \in L_2(N, \tau)$ is the GNS vector. The Stinespring space of θ_1 is

$$X^{\theta_1} = \left(\sum_i x_i \otimes \lambda(y_i) \right) L_2(M, tr) \otimes |1\rangle \cong M\mathbb{B} \subset \mathbb{B}(L_2(N, \tau), L_2(M, tr)).$$

The assumption that B is a partial isometry implies two things: 1) $M\mathcal{B}$ is a TRO in $\mathbb{B}(L_2(N, \tau), L_2(M, tr))$; 2) N is independent of $\mathcal{B}^*M\mathcal{B} \subset \mathbb{B}(L_2(N, \tau)$, the right algebra of $M\mathcal{B}$. Thus the assumption C1-3) implies that vN-channels are TRO channels. Note however, that in the vN-setup the dimensions of input and output system are always the same.

All examples from Section 8 of [15] are TRO channels. Thanks to the results from this paper, we now also obtain estimates for classical capacity, private capacity and their corresponding strong converse rates. In particular, Proposition 4.3 for high dimensional depolarizing channel also holds for the potential private capacity. That is, for a d -dimensional depolarizing channel $D_p(\rho) = q\rho + (1 - q)\frac{1}{d}$, we have

$$P^{(p)}(\mathcal{D}_q) \leq \log d - H\left(\frac{q(d^2 - 1) + 1}{d^2}\right) - \frac{(d^2 - 1)(1 - q)}{d^2} \log(d - 1).$$

5.2. New example. We illustrate how to find TRO channels by constructing a low dimensional example. Let $|\alpha| \leq 1$ be a real number. Define the channel $\Phi_\alpha : M_4 \rightarrow M_3$ as

follows,

$$\Phi_\alpha \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \right) = \begin{bmatrix} a_{11} + a_{22} & \alpha a_{13} & \alpha a_{24} \\ \alpha a_{31} & a_{33} & 0 \\ \alpha a_{42} & 0 & a_{44} \end{bmatrix}$$

This channel is nondegradable since it traces out the first 2×2 block. We will show that for this class of channels our estimates on quantum and private communication coincides and are actually tight,

$$Q^{(1)}(\Phi_\alpha) = Q^{(p)}(\Phi_\alpha) = Q^\dagger(\Phi_\alpha) = P^{(1)}(\Phi_\alpha) = P^{(p)}(\Phi_\alpha) = P^\dagger(\Phi_\alpha) = 1 - h\left(\frac{1+\alpha}{2}\right),$$

where $h(\lambda) = -\lambda \log \lambda - (1-\lambda) \log(1-\lambda)$ is the binary entropy function. Let us first consider the diagonal part of the channels. That is at $\alpha = 0$,

$$\Phi_0 \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \right) = \begin{bmatrix} a_{11} + a_{22} & 0 & 0 \\ 0 & a_{33} & 0 \\ 0 & 0 & a_{44} \end{bmatrix}$$

It is a direct sum of partial trace maps hence the Stinespring space corresponds to a TRO. The Stinespring isometry V_0 of Φ_0 is given by

$$V\left(\sum_i h_i e_i\right) = h_1 e_1 + h_2 e_1 \otimes e_2 + h_3 e_2 \otimes e_3 + h_4 e_3 \otimes e_4 \in H_3 \otimes H_4,$$

where $\{e_i\}$ is the computational basis. The corresponding operators are 3×4 matrices,

$$\widehat{Vh} = \begin{bmatrix} h_1 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{bmatrix}. \text{ Then the Stinespring space } X = M_{1,2} \oplus \mathbb{C} \oplus \mathbb{C} \text{ as a TRO. The}$$

left and right algebra are given by

$$B(X) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \quad E(X) = M_2 \oplus \mathbb{C} \oplus \mathbb{C}.$$

Let $S = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. One can verify that the only nontrivial $*$ -subalgebra independent of $E(X)$ in M_4 is

$$N = \{ \beta I + \alpha S \mid \alpha, \beta \in \mathbb{C} \}.$$

The normalized densities in N given by the one-parameter class $\{I + \alpha S \mid -1 \leq \alpha \leq 1\}$. Denote the symbol $f_\alpha = I + \alpha S$. Note that f_0 is the identity 1_E . Φ_α is a TRO channel

with symbol f_α ,

$$\Phi_\alpha(|h\rangle\langle h|) = \widehat{V}h f_\alpha \widehat{V}h^* = \begin{bmatrix} h_1 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \alpha \\ \alpha & 0 & 1 & 0 \\ 0 & \alpha & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{h}_1 & 0 & 0 \\ \bar{h}_2 & 0 & 0 \\ 0 & \bar{h}_3 & 0 \\ 0 & 0 & \bar{h}_4 \end{bmatrix}.$$

Via a change of basis, one can identify $f = \mathbb{C}I_2 \otimes \begin{bmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$. Thus for the entropy term we have $\tau(f \log f) = 1 - h(\frac{1+\alpha}{2})$. Since Φ_0 's outputs are all diagonal, then

$$Q^{(p)}(\Phi_0) = P^{(p)}(\Phi_0) = Q^\dagger(\Phi_0) = P^\dagger(\Phi_0) = 0.$$

We justify that $1 - h(\frac{1+\alpha}{2})$ is indeed an upper bound. On the other hand, $1 - h(\frac{1+\alpha}{2})$ is the quantum capacity of a qubit dephasing channel with parameter α which can be implement in Φ_α by using the block of $\{e_1, e_3\}$ (the first and the third basis vectors) or $\{e_2, e_4\}$. Hence our upper bound is also achievable.

5.2.1. A Physical Implementation. Far from being exotic, this channel is physically implementable with current technology, at least for real values of α . To show this, we will implement the channel using IBM's Quantum Experience, an online service that allows users to run simple quantum circuits (up to 5 qubits) on superconducting qubits at IBM facilities. While Quantum Experience supports the necessary qubit gates, it lacks the classically conditional logic and randomness generation that would allow us to directly program this channel. Therefore, we will show that this channel can be implemented as a classical mixture of simpler channels, which we will combine via post-selection and mixing of results from actual quantum circuits.

After a permutation of indices ($2 \leftrightarrow 4$, $1 \leftrightarrow 3$), we can see that the channel Φ_α is unitarily equivalent to

$$\Phi_\alpha \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & \alpha a_{12} & 0 & 0 \\ \alpha a_{21} & a_{22} + a_{44} & \alpha a_{43} & 0 \\ 0 & \alpha a_{34} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and will use this as the definition of the channel for implementation, as it is the simplest available. First, we show that this channel's output is a mixture of outputs from simpler channels.

$$\begin{bmatrix} a_{11} & \alpha a_{12} & 0 & 0 \\ \alpha a_{21} & a_{22} + a_{44} & \alpha a_{43} & 0 \\ 0 & \alpha a_{34} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & \alpha a_{12} & 0 & 0 \\ \alpha a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{44} & \alpha a_{43} & 0 \\ 0 & \alpha a_{34} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \alpha a_{12} & 0 & 0 \\ \alpha a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \text{SWAP}_{2 \leftrightarrow 4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & \alpha a_{43} \\ 0 & 0 & \alpha a_{34} & a_{44} \end{bmatrix}.$$

We can break down each of these components further:

$$\begin{bmatrix} a_{11} & \alpha a_{12} & 0 & 0 \\ \alpha a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \alpha \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and similarly for the other term. Therefore, we can write this channel's output as a classical mixture of the outputs of 4 component channels:

Name	Density Matrix	Circuit	Post-select for $q_1 = \dots$
$\rho_1 =$	$\begin{bmatrix} a_{11} & a_{21} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		$ 0\rangle$
$\rho_2 =$	$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		$ 0\rangle$
$\rho_3 =$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{44} & a_{43} & 0 \\ 0 & a_{34} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		$ 1\rangle$
$\rho_4 =$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{44} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		$ 1\rangle$

FIGURE 1. Images from IBM Quantum Experience showing gate implementations for the 4 constituent channels, where the upper qubit is q_2 and lower is q_1 . Since this implementation can't apply gates to already-measured qubits, we implement the conditionally applied CNOT gate by post-selecting on instances in which the CNOT was applied and q_1 was found to be 1, or in which it was not applied and q_1 was 0.

Let us consider these density matrices to be defined on a canonical 2-qubit basis,

$$|q_1, q_2\rangle \in \text{span}(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}) \quad (5.1)$$

such that q_1 is the outer index, in form $q_1 \otimes q_2$. 1st, whether we are in the 1-2 or 3-4 output subspace depends purely on the measurement of q_1 , which is completely dephased in every output. Therefore, we can essentially consider q_1 to be a classical bit for output purposes. We produce output ρ_1 by projecting to the $q_1 = |1\rangle$ subspace, and ρ_2 by additionally measuring q_2 within this subspace. We can implement ρ_3 and ρ_4 by similar steps, but followed by a CNOT gate with q_2 as the control and q_1 as the target bit. We can implement the channel for any α as

$$\Phi_\alpha(\rho) = \alpha(\rho_1 + \rho_2) + (1 - \alpha)(\rho_3 + \rho_4) \quad (5.2)$$

Since Quantum Experience does not have an efficient way of running the channel on a wide variety of states, we will simplify the input space by assuming that $q_1 = |0\rangle$ or $q_1 = |1\rangle$, allowing us to use the built-in tomography on q_2 . We know that this is a quantum capacity-achieving input subspace, so we do not lose much. Furthermore, we exploit the additivity of this channel to use the coherent information expression

$$Q(\Phi_\alpha) = Q^{(1)}(\Phi_\alpha) = I_c(A)B_{\Phi_\alpha(\rho)} = H(B)_{\Phi_\alpha(\rho)} - H(E)_{\Phi_\alpha(\rho)} = -H(A|B)_{\Phi_\alpha(\rho)} \quad (5.3)$$

For a pure state input without a reference system, $H(B) = H(E)$. We can simulate the effect of the reference system, which is traced out in the expression we will use, by mixing over results from these pure state inputs. We know that a capacity-achieving state could have the form

$$\rho = |0\rangle^{q_1} \otimes \frac{1}{\sqrt{2}}(|0\rangle^A \otimes |0\rangle^{q_2} + |1\rangle^A \otimes |1\rangle^{q_2}) \quad (5.4)$$

or equivalently with $|1\rangle^{q_1} \otimes \dots$. Therefore, we can simulate this state by mixing over input states with $q_2 = |0\rangle, |1\rangle$. We will also need to solve for a few other off-diagonal elements of the entangled density matrix, which we will achieve by implenting the channel on input states $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle + i|1\rangle)/\sqrt{2}$, and then using linear combinations of the implemented states to solve for the desired elements. This avoids explicitly simulating the reference system, and with a few other tricks, we are able to use IBM's built-in qubit tomography.

We will assume that q_1 is fixed to $|0\rangle$ or $|1\rangle$ for a given coding scheme, so it can't contribute any entropy anywhere. We are therefore only really interested in the state ρ as in equation 5.4, and its analog with $q_1 = 1$. Therefore, we can simulate the channel with

$\alpha = 1$ to be

$$\begin{aligned}
& CNOT(q_2 \rightarrow q_1) \cdot \Phi_1 \left(|1\rangle^{q_1} \otimes \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^{A \otimes q_2} \right) \approx \\
& |1\rangle^{q_1} \otimes \frac{1}{2} \begin{bmatrix} 0.977 & 0.003 + i0.006 & 0.060 & 0.964 + i0.082 \\ 0.003 - i0.006 & 0.023 & 0.004 - i0.018 & -0.060 \\ 0.060 & 0.004 + i0.018 & -0.0135 & -0.005 - i0.004 \\ 0.964 + i0.082 & -0.060 & -0.005 + i0.004 & 0.9865 \end{bmatrix}^{A \otimes q_2} \\
& \Phi_1 \left(|0\rangle^{q_1} \otimes \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^{A \otimes q_2} \right) \approx \\
& |0\rangle^{q_1} \otimes \frac{1}{2} \begin{bmatrix} 0.915 & -0.028 - i0.021 & -0.034 & 0.929 - i0.010 \\ -0.028 + i0.021 & 0.085 & 0.038 - i0.086 & 0.034 \\ -0.034 & 0.038 + i0.086 & -0.022 & -0.042 - i0.005 \\ 0.929 + i0.010 & 0.034 & -0.042 + i0.005 & 1.022 \end{bmatrix}^{A \otimes q_2}
\end{aligned}$$

We use an extra CNOT gate after the application of Φ_α when $q_1 = 1$ to undo the channel's CNOT gate (we can consider this extra to be in the decoder), which allows us to fully characterize the channel via single-qubit tomography. We can easily implement the trace in post-processing once we have characterized this output state. While we attempted to determine these states to 3 significant figures, we can see that this produces matrices that are slightly outside the set of densities due to realistic measurement errors. We can also simulate the channel for $\alpha = 0$ with full measurements, requiring no tomography. We will not use a reference system for this case either, since the complete dephasing should cause the reference system to become a classical copy of the input.

$$\begin{aligned}
& \cdot \Phi_1 \left(|0\rangle^{q_1} \otimes |0\rangle^{q_2} \right) \approx \begin{bmatrix} 0.823 & 0 & 0 & 0 \\ 0 & 0.092 & 0 & 0 \\ 0 & 0 & 0.076 & 0 \\ 0 & 0 & 0 & 0.009 \end{bmatrix} \\
& \cdot \Phi_1 \left(|0\rangle^{q_1} \otimes |1\rangle^{q_2} \right) \approx \begin{bmatrix} 0.082 & 0 & 0 & 0 \\ 0 & 0.800 & 0 & 0 \\ 0 & 0 & 0.008 & 0 \\ 0 & 0 & 0 & 0.109 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned} \cdot \Phi_1 \left(|1\rangle^{q_1} \otimes |0\rangle^{q_2} \right) &\approx \left[\begin{array}{c|c|c|c} 0.065 & 0 & 0 & 0 \\ 0 & 0.007 & 0 & 0 \\ 0 & 0 & 0.925 & 0 \\ 0 & 0 & 0 & 0.003 \end{array} \right] \\ \cdot \Phi_1 \left(|1\rangle^{q_1} \otimes |1\rangle^{q_2} \right) &\approx \left[\begin{array}{c|c|c|c} 0.037 & 0 & 0 & 0 \\ 0 & 0.937 & 0 & 0 \\ 0 & 0 & 0.007 & 0 \\ 0 & 0 & 0 & 0.019 \end{array} \right] \end{aligned}$$

Finally, we can put these together in the post-selection process.

$$\begin{aligned} &CNOT(q_2 \rightarrow q_1) \cdot \Phi_\alpha \left(|1\rangle^{q_1} \otimes \frac{1}{2} \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right]^{A \otimes q_2} \right) \approx \\ &|1\rangle^{q_1} \otimes \frac{\alpha}{2} \left[\begin{array}{c|c|c|c} 0.977 & 0.003 + i0.006 & 0.060 & 0.964 + i0.082 \\ \hline 0.003 - i0.006 & 0.023 & 0.004 - i0.018 & -0.060 \\ \hline 0.060 & 0.004 + i0.018 & -0.0135 & -0.005 - i0.004 \\ \hline 0.964 + i0.082 & -0.060 & -0.005 + i0.004 & 0.9865 \end{array} \right]^{A \otimes q_2} \\ &+ (1 - \alpha) \left(|1\rangle^{q_1} \otimes \left[\begin{array}{c|c|c|c} 0.466 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.471 \end{array} \right] + |0\rangle^{q_1} \otimes \left[\begin{array}{c|c|c|c} 0.051 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.011 \end{array} \right] \right) \\ &\Phi_\alpha \left(|0\rangle^{q_1} \otimes \frac{1}{2} \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right]^{A \otimes q_2} \right) \approx \\ &|0\rangle^{q_1} \otimes \frac{\alpha}{2} \left[\begin{array}{c|c|c|c} 0.915 & -0.028 - i0.021 & -0.034 & 0.929 - i0.010 \\ \hline -0.028 + i0.021 & 0.085 & 0.038 - i0.086 & 0.034 \\ \hline -0.034 & 0.038 + i0.086 & -0.022 & -0.042 - i0.005 \\ \hline 0.929 + i0.010 & 0.034 & -0.042 + i0.005 & 1.022 \end{array} \right]^{A \otimes q_2} \\ &+ (1 - \alpha) \left(|0\rangle^{q_1} \otimes \left[\begin{array}{c|c|c|c} 0.453 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.446 \end{array} \right] + |1\rangle^{q_1} \otimes \left[\begin{array}{c|c|c|c} 0.042 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.059 \end{array} \right] \right). \end{aligned}$$

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